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Connections between the basic and the Fibonacci-type series

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Let us denote the basic (original) Fibonacci series: u_n=1,1,2,3,5,...
We say that the a_n is a Fibonacci-type series, if a_1, a_2 are arbitrary natural numbers and $a_n = a_{n-1} + a_{n-2}$.

Theorem 1.
The connection of Fibonacci-type number and Fibonacci number is the above equality:

(1) $$a_n = a_1 \cdot u_{n-2} + a_2 \cdot u_{n-1}$$

Proof (mathematical induction):
If n=3 then $u_1 = 1,\quad u_2 = 1 \Rightarrow \quad a_3 = a_1 + a_2$ (it is true by the definition).
Assume a_n holds (for some unspecified value of *n*). It must then be shown that a_{n+1} holds, that is:

(2) $$a_{n+1} = a_n + a_{n-1} \xrightarrow{(1)} a_{n+1} = a_1 \cdot u_{n-2} + a_2 \cdot u_{n-1} + a_{n-1}$$

By the condition of induction:

(3) $$a_{n+1} = a_1 \cdot u_{n-2} + a_2 \cdot u_{n-1} + a_1 \cdot u_{n-3} + a_2 \cdot u_{n-2} = a_1(\underbrace{u_{n-2} + u_{n-3}}_{u_{n-1}}) + a_2(\underbrace{u_{n-1} + u_{n-2}}_{u_n})$$

Q.E.D.

Remark:
If $a_1 = a_2 = 1 \xrightarrow{(1)} a_n = u_{n-2} + u_{n-1} = u_n$, it is the basic Fibonacci series.

Example:
$a_1 = 111, a_2 = 222$ *then the* a_n *series* $\rightarrow$ 111, 222, 333, 555, 888, 1443, 2331, 3774 $\xrightarrow{(1)}$
$\xrightarrow{(1)} u_6 = 8,\ u_7 = 13 \Rightarrow a_8 = 111 \cdot 8 + 222 \cdot 13 = 3774$

Theorem 2.

(4) $$u_n = 1 + \sum_{i=1}^{n-2} u_i$$

Proof (mathematical induction):

If *n=3* then $u_3 = 1 + u_1 = 1 + 1 = 2$ (it is true by the definition).
Assume u_{n-1} holds (for some unspecified value of *n*). It must then be shown that u_n holds, that is:

$$u_n = u_{n-1} + u_{n-2} = 1 + \sum_{i=1}^{n-3} u_i + 1 + \sum_{j=1}^{n-4} u_j = 2 + 2\sum_{i=1}^{n-4} u_i \quad + u_{n-3} =$$

$$= 2\underbrace{(1 + \sum_{i=1}^{n-4} u_i)}_{u_{n-2}} + u_{n-3} = u_{n-2} + \underbrace{u_{n-2} + u_{n-3}}_{u_{n-1}} = u_n$$

Q.E.D.

Corollary:

If *n=m+d* *(d=1,2,3,...)* then by the Theorem 2.:

(5) $$u_n = u_{m+d} = 1 + \sum_{i=1}^{m+d-2} u_i = \underbrace{1 + \sum_{i=1}^{m-2} u_i}_{u_m} + \sum_{j=m-1}^{m+d-2} u_j = u_m + \sum_{j=m-1}^{n-2} u_j$$