Generalized Twin prime theorem

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Abstract

The *Symmetric prime number theorem* proof in [Dénes 2017] states, that there exists a symmetric prime pair (*p*, *q*) for any natural number *N≥4*, for which $p = N - m_N$ and $q = N + m_N$, and for which that is true $m_N = \frac{q-p}{2}$ and $N = \frac{q+p}{2}$.

Now we prove that for every m_N natural number there are infinite many symmetric prime pair (*Generalized Twin prime theorem*). Applied this proof for $m_N = 1$, we just got precisely the proof of the Twin prime conjecture, so thereafter we can called *Twin prime theorem*.

The proof of the basic theorem in this paper is based on the *Complementary Prime Sieve theorem* (see *CPS* in [Dénes 2001]). Due to this theorem, for any *N=6k+1* type natural number are composite iff one of the following is fulfilled: *k=6uv+u+v* or *k=6uv-u-v* (*u* and *v* are natural numbers). Based on this theorem, we prove with an indirect proof to the *Generalized Twin prime theorem*.

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LEMMA 1.

Any two prime numbers differ and their sum is even, so if *q>p>2* are prime numbers, that there are the $m_N = \frac{q-p}{2}$ and $N = \frac{q+p}{2}$ natural numbers.

Proof

Any prime number greater than 2 is odd. So, if

(1) $p = 2k + 1 \rangle\, 2$ prime number (*k* natural number)

(2) $q = 2l + 1 \rangle\, p$ prime number (*l* natural number)

then

(3) $$\frac{q-p}{2} = \frac{2l+1-2k-1}{2} = l + k$$

(4) $$\frac{q+p}{2} = \frac{2l+1+2k+1}{2} = l + k + 1$$

Q.E.D.

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From the Lema 1. follows the reverse of the *Dénes-type Symmetric prime number theorem* (see [Dénes 2017]), ie the following Theorem 1. is true.

THEOREM 1.

Let two prime numbers *q>p>2*, then there are $m_N = \frac{q-p}{2}$ and $N = \frac{q+p}{2}$ natural numbers, for which are valid $p = N - m_N$ and $q = N + m_N$.

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The *Symmetric prime number theorem* proven in [Dénes 2017] states, that there exists a symmetric prime pair (*p*, *q*) for any natural number *N≥4.*
Now, the question is: are there a $2m_N$ distence symmetric prime pair for every m_N natural number? This is proved in the following Theorem 2.

THEOREM 2.
For any m_N natural number there exists at least one *q*>*p* symmetric prime pair to which is fulfilled, that $q = p + 2m_N$.

Proof (indirect)
Suppose there is an $m_N = c$ natural number for which there is no *p* and *q* symmetric prime pair corresponding to the condition of this theorem. In this case, based on the *Complementary Prime Sieve theorem* (see Theorem 2. in [Dénes 2001]) for any prime number *p*, one of the following *q* natural numbers may be associated, because *q* is not a prime:

(5) $q = 6r - 1 = p + 2c$ and $r=6uv+u-v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots)$
(6) $q = 6r - 1 = p + 2c$ and $r=6uv-u+v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots)$
(7) $q = 6r + 1 = p + 2c$ and $r=6uv+u+v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots)$
(8) $q = 6r + 1 = p + 2c$ and $r=6uv-u-v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots)$

For the *p* prime we get the following formulas from the cases (5)-(8), one of which must be satisfied for every *u*, *v* natural number:

(9) $(5) \Rightarrow \quad p = 6r - 1 - 2c = 6(6uv + u - v) - 1 - 2c$
(10) $(6) \Rightarrow \quad p = 6r - 1 - 2c = 6(6uv - u + v) - 1 - 2c$
(11) $(7) \Rightarrow \quad p = 6r + 1 - 2c = 6(6uv + u + v) + 1 - 2c$
(12) $(8) \Rightarrow \quad p = 6r + 1 - 2c = 6(6uv - u - v) + 1 - 2c$

We show that if *u*=*v* then there exists a *u* natural number for which *p* in (9)-(12) is a composit number. But this contradicts the condition of the theorem, according to which *p* is a prime number.

(13)
$$\textit{If } u = v = \frac{c+1}{6}, \textit{ then } (9),(10) \Rightarrow$$
$$\Rightarrow p = 6(6u^2) - 1 - 2c = 6\left(6\frac{(c+1)^2}{6^2}\right) - 1 - 2c = (c+1)^2 - 1 - 2c = c^2$$

(14)
$$\textit{If } u = v = \frac{c-1}{6}, \textit{ then } (11) \Rightarrow$$
$$\Rightarrow p = 6(6u^2 + 2u) + 1 - 2c = 6\left(\frac{6(c-1)^2}{6^2} + \frac{2(c-1)}{6}\right) + 1 - 2c =$$
$$= (c-1)^2 + 2(c-1) + 1 - 2c = (c-1)^2 - 1 = (c-1-1)(c-1+1) = (c-2)c$$

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$$If\ \ u = v = \frac{c+1}{6}, then\ (12) \Rightarrow$$

(15) $$\Rightarrow\ p = 6(6u^2 - 2u) + 1 - 2c = 6\left(\frac{6(c+1)^2}{6^2} - \frac{2(c+1)}{6}\right) + 1 - 2c =$$

$$= (c+1)^2 - 2(c+1) + 1 - 2c = c^2 + 2c + 1 - 2c - 2 + 1 - 2c = c^2 - 2c = (c-2)c$$

The $u = v = \frac{c+1}{6}$ condition in (13) and (15) is always fulfilled if $c=6s-1$, and the $u = v = \frac{c-1}{6}$ condition in (14) is fulfilled if $c=6s+1$ ($s=1,2,3, \ldots$)

Q.E.D.

A few examples of the Theorem 2. are shown in Tables 1-5.

Table 1.

p		***q=p+4***
3		7
7		11
13		17
19		23
…		…
349		353
…		…
1.579		1.583
…		…
1.019.173		1.019.177
…		…
10.082.623		10.082.627
…		…
15.484.243		15.484.247
…		…

Table 2.

p		***q=p+6***
5		11
7		13
11		17
13		19
17		23
…		…
563		569
…		…
1.601		1.607
…		…
1.099.621		1.099.627
…		…
10.781.861		10.781.867
…		…
15.485.843		15.485.849
…		…

Table 3.

p		***q=p+8***
3		11
5		13
11		19
23		31
29		37
…		…
449		457
…		…
1.571		1.579
…		…
1.000.151		1.000.159
…		…
10.000.349		10.000.357
…		…
15.416.699		15.416.707
…		…

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Table 4.

p		*q=p+10*
3		13
7		17
13		23
19		29
…		…
73		83
…		…
433		443
…		…
751		761
…		…
1.153		1.163
…		…
10.000.759		10.000.769
…		…
13.985.341		13.985.351
…		…
15.484.549		15.484.559
…		…
444.333.973		444.333.983
…		…
888.889.501		888.889.511
…		…

Table 5.

p		*q=p+100*
3		103
7		107
13		113
31		131
…		…
487		587
…		…
1.723		1.823
…		…
1.000.033		1.000.133
…		…
10.000.591		10.000.691
…		…
15.485.341		15.485.441
…		…
444.333.313		444.333.413
…		…
888.889.501		888.889.601
…		…

Since $m_N=1$ for the p and $q=p+2m_N$ symmetric prime pairs are precisely the twin primes, so according to the Theorem 2. we can say the following Theorem 3. which is called *Generalized Twin prime theorem*.

THEOREM 3. (*Generalized Twin prime theorem*)

Let $q>p>2$ be symmetric prime pair with $2m_N$ distance, so that $m_N = \frac{q-p}{2}, N = \frac{q+p}{2}$, $p=N-m_N$ and $q=N+m_N$. Then there are infinite many p, q symmetric prime pairs for any m_N natural number.

Proof (indirect)

According to the above-proven Theorem 2. m_N can be any natural number.

Due to the Theorem 1. in [Dénes 2001] shown Table 6. below lists all natural numbers, so that columns 1. and 3. contain all the prime numbers.

Suppose that the *Kth* row is the last one in which p_K and $q_K = p_K + 2m_N$ are both prime numbers. In the rest of the proof of this theorem, for the shorter writing we will use the notation $m_N=c$.

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Table 6.

k	1. *6k-1* ↓	2. *6k*	3. *6k+1* ↓	4. *6k+2*	5. *6k+3*	6. *6k+4*
0			1	2	3	4
1	***5***	6	***7***	8	9	10
2	***11***	12	***13***	14	15	16
3	***17***	18	***19***	20	21	22
4	***23***	24	25	26	27	28
5	***29***	30	***31***	32	33	34
6	35	36	***37***	38	39	40
7	***41***	42	***43***	44	45	46
…	…	…	…	…	…	…
K	6K-1	6K	6K+1	6K+2	6K+3	6K+4
K+1	6(K+1)-1= 6K+5	6(K+1)= 6K+6	6(K+1)+1= 6K+7	6(K+1)+2= 6K+8	6(K+1)+3= 6K+9	6(K+1)+4= 6K+10
…	…	…	…	…	…	…
k=K+x	6k-1= 6(K+x)-1	6(K+x)	6k+1= 6(K+x)+1			
…	…	…	…	…	…	…

In the *Kth* row of Table 6. there are two prime numbers, so we have to examine the indirect conditions (16) and (17).

If $p_K = 6K - 1$ and $q_K = p_K + 2c$ are prime numbers, then for every *x* natural number:

(16)
$$\forall k = K + x \Rightarrow if \;\; p_k = 6(K+x) - 1 \; is \;\; prime, then \quad q_k = p_k + 2c =$$
$$= 6(K+x) - 1 + 2c = \underbrace{6K - 1 + 2c}_{q_K} + 6x = q_K + 6x \; is \;\; NOT \;\; prime$$

If $p_K = 6K + 1$ and $q_K = p_K + 2c$ are prime numbers, then for every *x* natural number:

(17)
$$\forall k = K + x \Rightarrow if \;\; p_k = 6(K+x) + 1 \; is \;\; prime, then \quad q_k = p_k + 2c =$$
$$= 6(K+x) + 1 + 2c = \underbrace{6K + 1 + 2c}_{q_K} + 6x = q_K + 6x \; is \;\; NOT \;\; prime$$

Due to the indirect condition q_k is not a prime, so from the deductions (16) and (17) follows that any *u*, *v* natural numbers has one of the connections (5)-(8) exists. It follows that we get the following relationships for q_k and q_K.

(18)
$$q_k \overset{(5)}{=} 6r - 1 = q_K + 6x \quad \text{and} \;\; r{=}6uv{+}u{-}v \quad (u{=}1,2,3,\ldots), (v{=}1,2,3,\ldots) \Rightarrow$$
$$\Rightarrow q_K = 6r - 1 - 6x = 6(6uv + u - v) - 1 - 6x$$

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(19) $q_k \overset{(6)}{=} 6r-1 = q_K + 6x$ and $r=6uv-u+v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots) \Rightarrow$

$$\Rightarrow q_K = 6r-1-6x = 6(6uv-u+v)-1-6x$$

(20) $q_k \overset{(7)}{=} 6r+1 = q_K + 6x$ and $r=6uv+u+v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots) \Rightarrow$

$$\Rightarrow q_K = 6r+1-6x = 6(6uv+u+v)+1-6x$$

(21) $q_k \overset{(8)}{=} 6r+1 = q_K + 6x$ and $r=6uv-u-v$ $(u=1,2,3,\ldots), (v=1,2,3,\ldots) \Rightarrow$

$$\Rightarrow q_K = 6r+1-6x = 6(6uv-u-v)+1-6x$$

Due to the indirect conditions (16)-(17) anyway we choose the *u*, *v* natural numbers the q_K is prime. Now we show that in the case of *u*=*v*, each of the cases (18)-(21) has infinite number of *x* values for which q_K a composite number and this contradicts the indirect conditions.

(22)
$$u=v \overset{(18),(19)}{\Rightarrow} q_K = 6(6u^2)-1-6x \quad and \quad u=\frac{x+1}{6} \Rightarrow q_K = 6\frac{6(x+1)^2}{6^2}-1-6x =$$
$$= x^2+2x+1-1-6x = x^2-4x = x(x-4)$$

Since *u* is a natural number then the condition $u=\frac{x+1}{6}$ is always true if *x*=*6l-1*, where *l* is a natural number (hence *x* = *5*,*11*,*17*,*23*, ...).

(23)
$$u=v \overset{(20)}{\Rightarrow} q_K = 6(6u^2+2u)+1-6x \quad and \quad u=\frac{x-1}{6} \Rightarrow$$
$$\Rightarrow q_K = 6\left(\frac{6(x-1)^2}{6^2}+\frac{2(x-1)}{6}\right)+1-6x = (x-1)^2+2(x-1)+1-6x = x(x-6)$$

(24)
$$u=v \overset{(21)}{\Rightarrow} q_K = 6(6u^2-2u)+1-6x \quad and \quad u=\frac{x+1}{6} \Rightarrow$$
$$\Rightarrow q_K = 6\left(\frac{6(x+1)^2}{6^2}-\frac{2(x+1)}{6}\right)+1-6x = (x+1)^2-2(2+1)+1-6x = x(x-6)$$

Since *u* is a natural number then the condition $u=\frac{x-1}{6}$ is always true if *x*=*6l+1*, where *l* is a natural number (hence *x* = *7*,*13*,*19*,*25*, ...).

Q.E.D.

It is clear that if Theorem 3. applied for $m_N=1$, it is precisely the proof of the classic Twin prime conjecture, so thereafter we can called *Twin prime theorem.*

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References

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