

Proof of the existence of prime number between successive squares

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Abstract

In the present paper, we prove that for any n natural number, in the $[n^2, (n+1)^2]$ interval there are always exist $6k \pm 1$ type g_1, g_2, \dots, g_t natural numbers. We give a closed formula to the n -dependent value t and on the g_1, g_2, \dots, g_t natural numbers. After all, based on the *Dénes-type Complementary Prime-sieve theorem* (see [Dénes 2001]), we prove that among the g_1, g_2, \dots, g_t natural numbers for any n , there are at least one prime number.

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THEOREM 1.

Let n be any natural number, then there always exist a q prime number on which it is fulfilled

$$(1) \quad n^2 \langle q \langle (n+1)^2$$

Proof

Let denote $p_1, p_2, \dots, p_i, \dots, p_r$ the all prime numbers less than n^2

$$(2) \quad p_1 \langle p_2 \langle \dots \langle p_i \langle \dots \langle p_r \langle n^2$$

Since $(n+1)^2 - n^2 = 2n+1$ ($n \geq 2$) then in the $[n^2, (n+1)^2]$ interval has at least one natural number in the form $g_x = 6z \pm 1$, where $x=1, 2, 3, \dots, t$ és $z=1, 2, 3, \dots$ Search for the first of these type (g_1) numbers after n^2 (see Figure 1).

$$(3a) \quad n = 2k \text{ and } k = 3s \ (s = 1, 2, 3, \dots) \Rightarrow n^2 = (2 \cdot 3s)^2 = 6^2 \cdot s^2 \Rightarrow g_1 = n^2 + 1 \rightarrow (6z + 1)$$

$$(3b) \quad \begin{aligned} n = 2k \text{ and } k = 3s - 1 \ (s = 1, 2, 3, \dots) &\Rightarrow \\ \Rightarrow n^2 = (2(3s - 1))^2 = 4(3s - 1)^2 = 4(9s^2 - 6s + 1) = 36s^2 - 24s + 4 = \\ = 6(6s^2 - 4s) + 4 &\Rightarrow g_1 = n^2 + 1 \rightarrow (6z - 1) \end{aligned}$$

$$(3c) \quad \begin{aligned} n = 2k \text{ and } k = 3s - 2 \ (s = 1, 2, 3, \dots) &\Rightarrow \\ \Rightarrow n^2 = (2(3s - 2))^2 = 4(3s - 2)^2 = 4(9s^2 - 12s + 4) = 36s^2 - 48s + 16 = \\ = 6(6s^2 - 8s + 2) + 4 &\Rightarrow g_1 = n^2 + 1 \rightarrow (6z - 1) \end{aligned}$$

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$$(3d) \quad n = 2k + 1 \text{ and } k = 3s \ (s = 1, 2, 3, \dots) \Rightarrow n^2 = (2 \cdot 3s + 1)^2 = 36s^2 + 12s + 1 = \\ = 6(6s^2 + 2s) + 1 \Rightarrow g_1 = n^2 + 4 \rightarrow (6z - 1)$$

$$(3e) \quad n = 2k + 1 \text{ and } k = 3s - 1 \ (s = 1, 2, 3, \dots) \Rightarrow \\ \Rightarrow n^2 = (2(3s - 1) + 1)^2 = 4(3s - 1)^2 + 4(3s - 1) + 1 = \\ = 4(9s^2 - 6s + 1) + 12s - 4 + 1 = 36s^2 - 24s + 4 + 12s - 4 + 1 = 36s^2 - 12s + 1 = \\ = 6(6s^2 - 2s) + 1 \Rightarrow g_1 = n^2 + 4 \rightarrow (6z - 1)$$

$$(3f) \quad n = 2k + 1 \text{ and } k = 3s - 2 \ (s = 1, 2, 3, \dots) \Rightarrow \\ \Rightarrow n^2 = (2(3s - 2) + 1)^2 = 4(3s - 2)^2 + 4(3s - 2) + 1 = \\ = 4(9s^2 - 12s + 4) + 12s - 8 + 1 = 36s^2 - 48s + 16 + 12s - 8 + 1 = 36s^2 - 36s + 6 + 3 = \\ = 6(6s^2 - 6s + 1) + 3 \Rightarrow g_1 = n^2 + 2 \rightarrow (6z - 1)$$

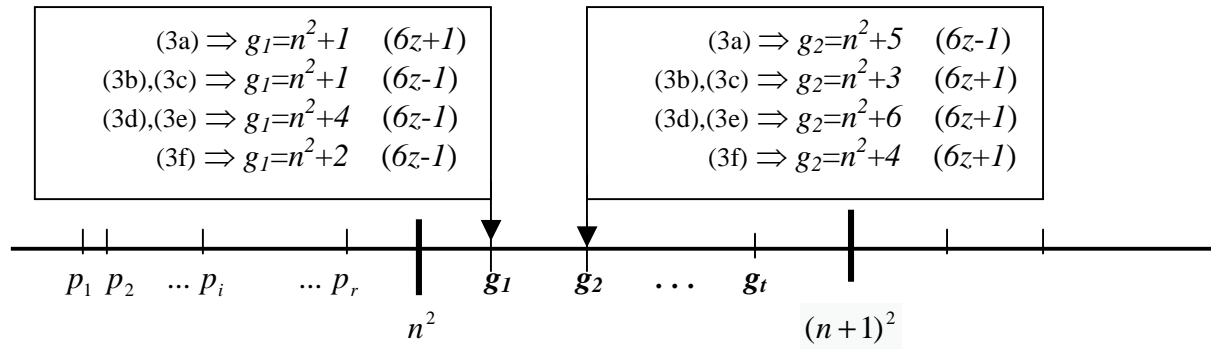


Figure 1.

According to the (3a)-(3f) formulas, g_1 is $6z+1$ type only in the case (3a), in all other cases are there of $6z-1$ type. Since we search the $g_x=6z\pm 1$ type natural numbers in the $[n^2, (n+1)^2]$ interval, then there exists in each case a g_2 number that is opposite type to g_1 . So on g_2 are true of the following relationships:

$$(4a) \quad \stackrel{(3a)}{\Rightarrow} g_1 = n^2 + 1 \rightarrow (6z + 1) \Rightarrow g_2 \rightarrow (6z - 1) \Rightarrow g_2 = g_1 + 4 = n^2 + 5$$

$$(4b) \quad \stackrel{(3b)}{\Rightarrow} g_1 = n^2 + 1 \rightarrow (6z - 1) \Rightarrow g_2 \rightarrow (6z + 1) \Rightarrow g_2 = g_1 + 2 = n^2 + 3$$

$$(4c) \quad \stackrel{(3c)}{\Rightarrow} g_1 = n^2 + 1 \rightarrow (6z - 1) \Rightarrow g_2 \rightarrow (6z + 1) \Rightarrow g_2 = g_1 + 2 = n^2 + 3$$

$$(4d) \quad \stackrel{(3d)}{\Rightarrow} g_1 = n^2 + 4 \rightarrow (6z - 1) \Rightarrow g_2 \rightarrow (6z + 1) \Rightarrow g_2 = g_1 + 2 = n^2 + 6$$

$$(4e) \quad \stackrel{(3e)}{\Rightarrow} g_1 = n^2 + 4 \rightarrow (6z-1) \Rightarrow g_2 \rightarrow (6z+1) \Rightarrow g_2 = g_1 + 2 = n^2 + 6$$

$$(4f) \quad \stackrel{(3f)}{\Rightarrow} g_1 = n^2 + 2 \rightarrow (6z-1) \Rightarrow g_2 \rightarrow (6z+1) \Rightarrow g_2 = g_1 + 2 = n^2 + 4$$

Table 1. presents some examples to calculating g_1 and g_2 values. Column 4 of the Table 1. shows the calculation of g_1 and column 7 shows the calculation of g_2 with the formulas (3a)-(3f) and (4a)-(4f). Columns 6 and 9 contain the values g_1 and g_2 (**bold highlighting** indicates that g_1 or g_2 is prime).

We call g_1 -type the next arithmetic series in the $[n^2, (n+1)^2]$ interval, and denote t_1 the number of elements of the series:

$$(5) \quad g_1, g_1+6, g_1+2 \cdot 6, \dots, g_1+(t_1-1) \cdot 6$$

Using the formulas (3a)-(3f) we can determine the number of elements of the g_1 -type series (t_1).

$$(6a) \quad \begin{aligned} &\stackrel{(3a)}{\Rightarrow} g_1 = n^2 + 1 \Rightarrow \\ \Rightarrow t_1 &= \left[\frac{(n+1)^2 - (n^2 + 1)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 1 + 3}{6} \right] = \left[\frac{2n + 3}{6} \right] \end{aligned}$$

$$(6b) \quad \begin{aligned} &\stackrel{(3b)}{\Rightarrow} g_1 = n^2 + 1 \Rightarrow \\ \Rightarrow t_1 &= \left[\frac{(n+1)^2 - (n^2 + 1)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 1 + 3}{6} \right] = \left[\frac{2n + 3}{6} \right] \end{aligned}$$

$$(6c) \quad \begin{aligned} &\stackrel{(3c)}{\Rightarrow} g_1 = n^2 + 1 \Rightarrow \\ \Rightarrow t_1 &= \left[\frac{(n+1)^2 - (n^2 + 1)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 1 + 3}{6} \right] = \left[\frac{2n + 3}{6} \right] \end{aligned}$$

$$(6d) \quad \begin{aligned} &\stackrel{(3d)}{\Rightarrow} g_1 = n^2 + 4 \Rightarrow \\ \Rightarrow t_1 &= \left[\frac{(n+1)^2 - (n^2 + 4)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 4 + 3}{6} \right] = \left[\frac{n}{3} \right] \end{aligned}$$

$$(6e) \quad \begin{aligned} &\stackrel{(3e)}{\Rightarrow} g_1 = n^2 + 4 \Rightarrow \\ \Rightarrow t_1 &= \left[\frac{(n+1)^2 - (n^2 + 4)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 4 + 3}{6} \right] = \left[\frac{n}{3} \right] \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(3f)}{\Rightarrow} g_1 = n^2 + 2 \Rightarrow \\
 (6f) \quad & \Rightarrow t_1 = \left[\frac{(n+1)^2 - (n^2 + 2)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 2 + 3}{6} \right] = \left[\frac{2n + 2}{6} \right] = \left[\frac{n+1}{3} \right]
 \end{aligned}$$

We call g_2 -type the next arithmetic series in the $[n^2, (n+1)^2]$ interval, and denote t_2 the number of elements of the series:

$$(7) \quad g_2, g_2+6, g_2+2 \cdot 6, \dots, g_2+(t_2-1) \cdot 6$$

Using the formulas (4a)-(4f) we can determine the number of elements of the g_2 -type series (t_2).

$$\begin{aligned}
 & \stackrel{(4a)}{\Rightarrow} g_2 = n^2 + 5 \Rightarrow \\
 (8a) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 5)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 5 + 3}{6} \right] = \left[\frac{2n-1}{6} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4b)}{\Rightarrow} g_2 = n^2 + 3 \Rightarrow \\
 (8b) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 3)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 3 + 3}{6} \right] = \left[\frac{2n+1}{6} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4c)}{\Rightarrow} g_2 = n^2 + 3 \Rightarrow \\
 (8c) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 3)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 3 + 3}{6} \right] = \left[\frac{2n+1}{6} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4d)}{\Rightarrow} g_2 = n^2 + 6 \Rightarrow \\
 (8d) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 6)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 6 + 3}{6} \right] = \left[\frac{n-1}{3} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4e)}{\Rightarrow} g_2 = n^2 + 6 \Rightarrow \\
 (8e) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 6)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 6 + 3}{6} \right] = \left[\frac{n-1}{3} \right]
 \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(4f)}{\Rightarrow} g_2 = n^2 + 4 \Rightarrow \\
 (8f) \quad & \Rightarrow t_2 = \left[\frac{(n+1)^2 - (n^2 + 4)}{6} + \frac{1}{2} \right] = \left[\frac{n^2 + 2n + 1 - n^2 - 4 + 3}{6} \right] = \left[\frac{n}{3} \right]
 \end{aligned}$$

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We are obtained the number t of the $g_x=6z\pm 1$ -type elements in the $[n^2, (n+1)^2]$ interval (see figure 1.) by summing the pairwise t_1 and t_2 (see (6a)-(6f), (8a)-(8f) and Table 1. column 10).

$$(9a) \quad \stackrel{(6a),(8a)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{2n+3}{6} \right] + \left[\frac{2n-1}{6} \right] = \left[\frac{2n+1}{3} \right]$$

$$(9b) \quad \stackrel{(6b),(8b)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{2n+3}{6} \right] + \left[\frac{2n+1}{6} \right] = \left[\frac{2n+2}{3} \right]$$

$$(9c) \quad \stackrel{(6c),(8c)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{2n+3}{6} \right] + \left[\frac{2n+1}{6} \right] = \left[\frac{2n+2}{3} \right]$$

$$(9d) \quad \stackrel{(6d),(8d)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{n}{3} \right] + \left[\frac{n-1}{3} \right] = \left[\frac{2n-1}{3} \right]$$

$$(9e) \quad \stackrel{(6e),(8e)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{n}{3} \right] + \left[\frac{n-1}{3} \right] = \left[\frac{2n-1}{3} \right]$$

$$(9f) \quad \stackrel{(6f),(8f)}{\Rightarrow} \quad t = t_1 + t_2 = \left[\frac{n+1}{3} \right] + \left[\frac{n}{3} \right] = \left[\frac{2n+1}{3} \right]$$

It follows from the above and from the Theorem 1. in [Dénes 2001] that there are always t pieces of $6z\pm 1$ -type g_1, \dots, g_t values (see (9a)-(9f)) in the $[n^2, (n+1)^2]$ interval and only among them there exists the prime numbers in the intervals. Cases $n=3,4, \dots, 10$ are shown in Figures 2-9. In the Figures the arrows indicate the corresponding g_i values, with the **bold arrows** marked if g_i is a prime number.

Examples

$$n = 3 \stackrel{(9f)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 3 + 1}{3} \right\rfloor = 2$$

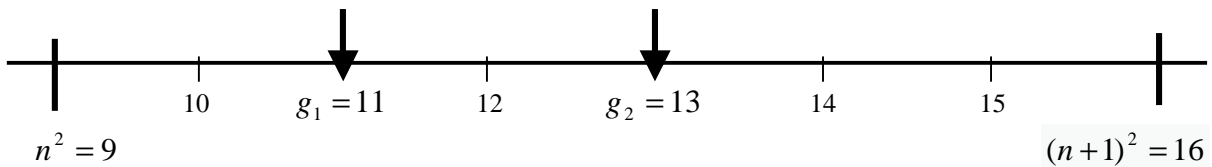


Figure 2.

$$n = 4 \stackrel{(9b)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 4 + 2}{3} \right\rfloor = 3$$

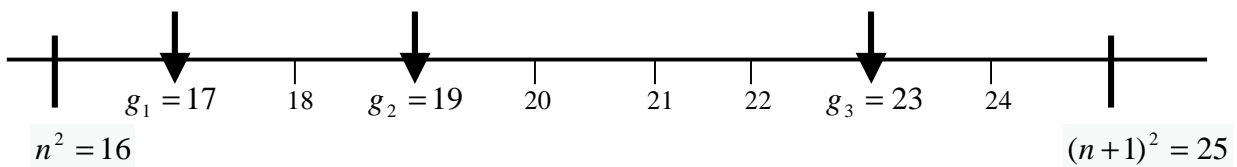


Figure 3.

$$n = 5 \stackrel{(9e)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 5 - 1}{3} \right\rfloor = 3$$

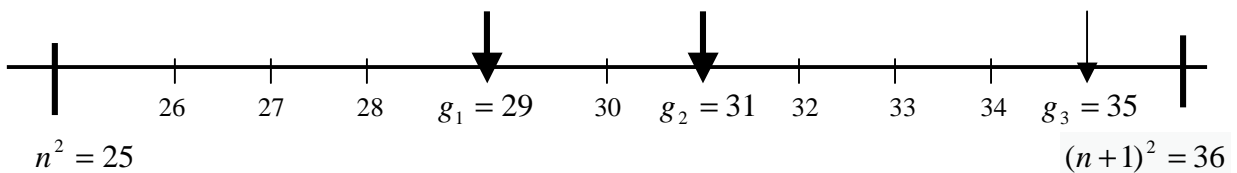


Figure 4.

$$n = 6 \stackrel{(9a)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 6 + 1}{3} \right\rfloor = 4$$

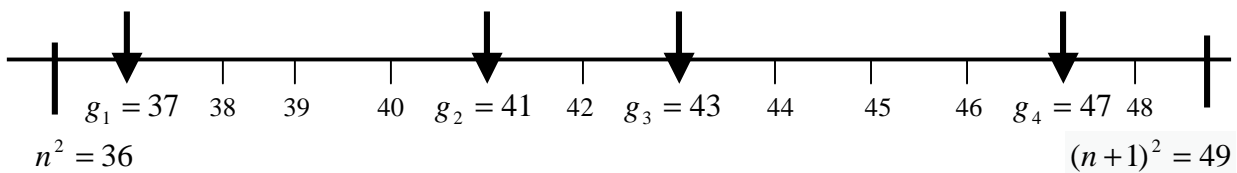


Figure 5.

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$$n = 7 \stackrel{(9d)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 7 - 1}{3} \right\rfloor = 4$$

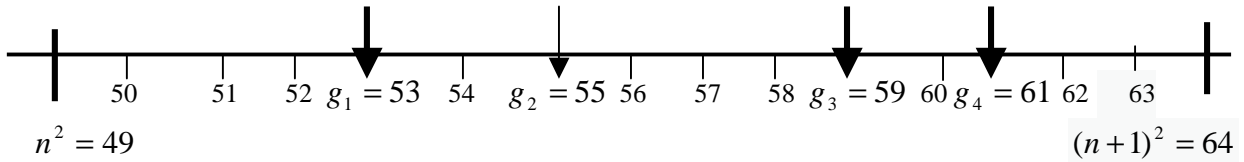


Figure 6.

$$n = 8 \stackrel{(9c)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 8 + 2}{3} \right\rfloor = 6$$

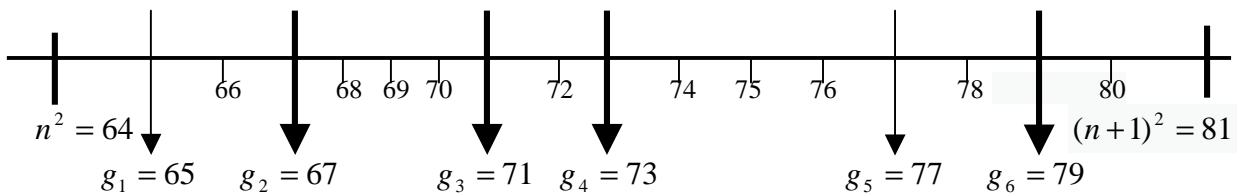


Figure 7.

$$n = 9 \stackrel{(9f)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 9 + 1}{3} \right\rfloor = 6$$

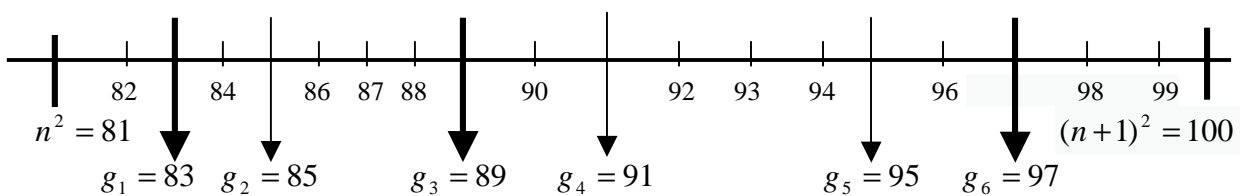


Figure 8.

$$n = 10 \stackrel{(9b)}{\Rightarrow} t = \left\lfloor \frac{2 \cdot 10 + 2}{3} \right\rfloor = 7$$

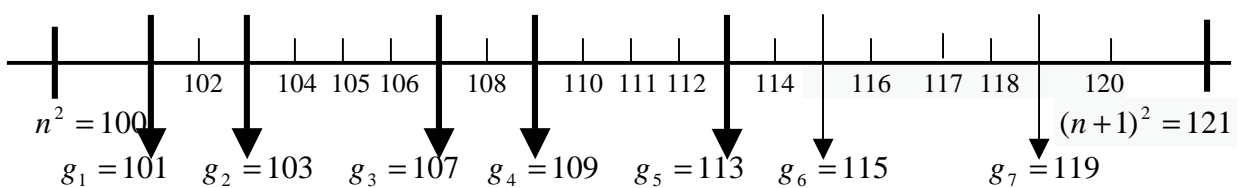


Figure 9.

Continue of the proof (indirect)

As an indirect condition contrary to statement (1), let us suppose that the g_1, \dots, g_t -types values (see (5), (7)) for any natural number n is not a prime number. Therefore, based on the formulas (3a)-(3f), due to [Dénes 2001] Theorem 2, to each g_x ($x=1, 2, \dots, t$) must be true one of the following equations:

$$(10a) \quad \forall n \text{ és } g_x = 6z + 1 \quad (x = 1, 2, \dots, t) \Rightarrow z = 6uv + u + v \quad (u, v = 1, 2, 3, \dots)$$

$$(10b) \quad \forall n \text{ és } g_x = 6z + 1 \quad (x = 1, 2, \dots, t) \Rightarrow z = 6uv - u - v \quad (u, v = 1, 2, 3, \dots)$$

$$(10c) \quad \forall n \text{ és } g_x = 6z - 1 \quad (x = 1, 2, \dots, t) \Rightarrow z = 6uv + u - v \quad (u, v = 1, 2, 3, \dots)$$

$$(10d) \quad \forall n \text{ és } g_x = 6z - 1 \quad (x = 1, 2, \dots, t) \Rightarrow z = 6uv - u + v \quad (u, v = 1, 2, 3, \dots)$$

We show that in the case of each g_x values ($1 \leq x \leq t$) for all n natural numbers (see (3a)-(3f) and (5), (7), as well as Figure 1.), the indirect statement always leads to a contradiction.

Due to the formulas (5) and (7) the values g_x ($x=1, 2, \dots, t$) can be sorted into the following series:

$$(11) \quad g_1, g_2, g_3 = g_1 + 6, g_4 = g_2 + 6, \dots, g_x = g_1 + \left[\frac{x-1}{2} \right] 6, g_{x+1} = g_2 + \left[\frac{x-2}{2} \right] 6, \dots, g_t \Rightarrow \\ \Rightarrow g_1, g_2, g_3 = g_1 + 6, g_4 = g_2 + 6, \dots, g_x = g_1 + 3(x-1), g_{x+1} = g_2 + 3(x-2), \dots, g_t$$

For the rest of the proof, we will calculate all the pairs of formulas (3a)-(3f) and (10a)-(10d), which produce all n natural numbers (this is $6 \cdot 4 = 24$ cases). In other words, we produce the formulas to the g_x and g_{x+l} elements of the g_1, \dots, g_t series due to the (10a)-(10d) correspondence.

Case of the (3a) - (10a) – (10c):

$$(12a) \quad g_x = g_1 + 3(x-1) \stackrel{(3a)}{=} n^2 + 1 + 3(x-1) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3a)}{=} n^2 + 5 + 3(x-2) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$\stackrel{(12a)}{\Rightarrow} \frac{n^2 + 3(x-1)}{6} = v(6u+1) + u \Rightarrow v = \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} \stackrel{(12a)}{\Rightarrow}$$

$$\stackrel{(12a)}{\Rightarrow} \frac{n^2 + 6 + 3(x-2)}{6} = 6uv + u - v \Rightarrow$$

$$\Rightarrow \frac{n^2 + 6 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} + u - \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} =$$

$$(12b) \quad = \frac{6un^2 + 18u(x-1) - 36u^2 + 36u^2 + 6u - n^2 - 3(x-1) + 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 12u - n^2 - 3(x-1)}{6(6u+1)} \Rightarrow$$

$$\Rightarrow (n^2 + 6 + 3(x-2))(6u+1) = 6un^2 + 18u(x-1) + 12u - n^2 - 3(x-1) \Rightarrow$$

$$\Rightarrow 6un^2 + 18ux + n^2 + 3x = 6un^2 + 18ux - 6u - n^2 - 3x + 3 \Rightarrow$$

$$\Rightarrow 2n^2 + 6x + 6u - 3 = 0 \Rightarrow u = \frac{-2n^2 - 6x + 3}{6} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (12a)-(12b) is contrary!

Case of the (3a) - (10a) – (10d):

$$(13a) \quad g_x = g_1 + 3(x-1) \stackrel{(3a)}{=} n^2 + 1 + 3(x-1) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3a)}{=} n^2 + 5 + 3(x-2) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$\stackrel{(13a)}{\Rightarrow} \frac{n^2 + 3(x-1)}{6} = v(6u+1) + u \Rightarrow v = \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} \stackrel{(13a)}{\Rightarrow}$$

$$\stackrel{(13a)}{\Rightarrow} \frac{n^2 + 6 + 3(x-2)}{6} = 6uv - u + v \Rightarrow$$

$$\Rightarrow \frac{n^2 + 6 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} - u + \frac{n^2 + 3(x-1) - 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18u(x-1) - 36u^2 - 36u^2 - 6u + n^2 + 3(x-1) - 6u}{6(6u+1)} =$$

$$(13b) \quad = \frac{6un^2 + 18u(x-1) - 72u^2 - 12u + n^2 + 3(x-1)}{6(6u+1)} \Rightarrow$$

$$\Rightarrow (n^2 + 6 + 3(x-2))(6u+1) = 6un^2 + 18ux - 30u - 72u^2 + n^2 + 3x - 3 \Rightarrow$$

$$\Rightarrow 6un^2 + 18ux + n^2 + 3x = 6un^2 + 18ux - 30u - 72u^2 + n^2 + 3x - 3 \Rightarrow$$

$$\Rightarrow 72u^2 + 30u + 3 = 0 \Rightarrow 24u^2 + 10u + 1 = 0 \Rightarrow$$

$$\Rightarrow u_{1,2} = \frac{-10 \pm \sqrt{100 - 96}}{48} = \frac{-10 \pm 2}{48} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (13a)-(13b) is contrary!

Case of the (3a) - (10b) – (10c):

$$(14a) \quad g_x = g_1 + 3(x-1) \stackrel{(3a)}{=} n^2 + 1 + 3(x-1) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3a)}{=} n^2 + 5 + 3(x-2) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$\stackrel{(14a)}{\Rightarrow} \frac{n^2 + 3(x-1)}{6} = v(6u-1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} \stackrel{(14a)}{\Rightarrow}$$

$$\stackrel{(14a)}{\Rightarrow} \frac{n^2 + 6 + 3(x-2)}{6} = 6uv + u - v \Rightarrow$$

$$\Rightarrow \frac{n^2 + 6 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} + u - \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 36u^2 + 36u^2 - 6u - n^2 - 3(x-1) - 6u}{6(6u-1)} =$$

$$(14b) \quad = \frac{6un^2 + 18u(x-1) + 72u^2 - 12u - n^2 - 3(x-1)}{6(6u-1)} \Rightarrow$$

$$\Rightarrow (n^2 + 6 + 3(x-2))(6u-1) = 6un^2 + 18ux - 30u + 72u^2 - n^2 - 3x + 3 \Rightarrow$$

$$\Rightarrow 6un^2 + 18ux - n^2 - 3x = 6un^2 + 18ux - 30u + 72u^2 - n^2 - 3x + 3 \Rightarrow$$

$$\Rightarrow 72u^2 - 30u + 3 = 0 \Rightarrow 24u^2 - 10u + 1 = 0 \Rightarrow$$

$$\Rightarrow u_{1,2} = \frac{10 \pm \sqrt{100 - 96}}{48} = \frac{10 \pm 2}{48} < 1$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (14a)-(14b) is contrary!

Case of the (3a) - (10b) – (10d):

$$(15a) \quad g_x = g_1 + 3(x-1) \stackrel{(3a)}{=} n^2 + 1 + 3(x-1) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3a)}{=} n^2 + 5 + 3(x-2) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$\stackrel{(15a)}{\Rightarrow} \frac{n^2 + 3(x-1)}{6} = v(6u-1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} \stackrel{(15a)}{\Rightarrow}$$

$$\stackrel{(15a)}{\Rightarrow} \frac{n^2 + 6 + 3(x-2)}{6} = 6uv - u + v \Rightarrow$$

$$\Rightarrow \frac{n^2 + 6 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} - u + \frac{n^2 + 3(x-1) + 6u}{6(6u-1)} =$$

$$(15b) \quad = \frac{6un^2 + 18ux - 6u + n^2 + 3x - 3}{6(6u-1)} \Rightarrow$$

$$\Rightarrow (n^2 + 6 + 3(x-2))(6u-1) = 6un^2 + 18ux - 6u + n^2 + 3x - 3 \Rightarrow$$

$$\Rightarrow 6un^2 + 18ux - n^2 - 3x = 6un^2 + 18ux - 6u + n^2 + 3x - 3 \Rightarrow$$

$$\Rightarrow 6u = 2n^2 + 6x - 3 \Rightarrow u = \frac{2n^2 + 6x - 3}{6} = x + \underbrace{\frac{n^2}{3} - \frac{1}{2}}_{\text{never an integer}}$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (15a)-(15b) is contrary!

Case of the (3b),(3c) - (10c) - (10a):

$$(16a) \quad g_x = g_1 + 3(x-1) \stackrel{(3b),(3c)}{=} n^2 + 1 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3b),(3c)}{=} n^2 + 3 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$\stackrel{(16a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 2}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} \stackrel{(16a)}{\Rightarrow}$$

$$\stackrel{(16a)}{\Rightarrow} \frac{n^2 + 2 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(16b) \quad \Rightarrow \frac{n^2 + 2 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} + u + \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 12u - 36u^2 + 36u^2 - 6u + n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux - 12u + 3x + n^2 - 1}{6(6u-1)}$$

$$\stackrel{(16b)}{\Rightarrow} (n^2 + 2 + 3(x-2))(6u-1) = 6un^2 + 18ux - 12u + 3x + n^2 - 1 \Rightarrow$$

$$(16c) \quad \Rightarrow 6un^2 + 18ux - 24u - n^2 - 3x + 4 = 6un^2 + 18ux - 12u + 3x + n^2 - 1 \Rightarrow$$

$$\Rightarrow 2n^2 + 12u + 3x - 3 = 0 \Rightarrow u = \frac{3 - 2n^2 - 3x}{12} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (16a)-(16c) is contrary!

Case of the (3b),(3c) - (10c) - (10b):

$$(17a) \quad g_x = g_1 + 3(x-1) \stackrel{(3b),(3c)}{=} n^2 + 1 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3b),(3c)}{=} n^2 + 3 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(17a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 2}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} \stackrel{(17a)}{\Rightarrow}$$

$$\stackrel{(17a)}{\Rightarrow} \frac{n^2 + 2 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(17b) \quad \Rightarrow \frac{n^2 + 2 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} - u - \frac{n^2 + 3(x-1) + 2 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 12u - 36u^2 - 36u^2 + 6u - n^2 - 3(x-1) - 2 + 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux + 6u - 72u^2 - 3x - n^2 + 1}{6(6u-1)}$$

$$\stackrel{(17b)}{\Rightarrow} (n^2 + 2 + 3(x-2))(6u-1) = 6un^2 + 18ux + 6u - 72u^2 - 3x - n^2 + 1 \Rightarrow$$

$$(17c) \quad \Rightarrow 6un^2 + 18ux - 24u - n^2 - 3x + 4 = 6un^2 + 18ux + 6u - 72u^2 - 3x - n^2 + 1 \Rightarrow$$

$$\Rightarrow 72u^2 - 30u - 1 = 0 \Rightarrow u_{1,2} = \frac{30 \pm \sqrt{30^2 + 4 \cdot 72}}{2 \cdot 72} = \frac{15 \pm \sqrt{15^2 + 72}}{72}$$

not an integer

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (17a)-(17c) is contrary!

Case of the (3b),(3c) - (10d) - (10a):

$$(18a) \quad g_x = g_1 + 3(x-1) \stackrel{(3b),(3c)}{=} n^2 + 1 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3b),(3c)}{=} n^2 + 3 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$\stackrel{(18a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 2}{6} = v(6u + 1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} \stackrel{(18a)}{\Rightarrow}$$

$$\stackrel{(18a)}{\Rightarrow} \frac{n^2 + 2 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(18b) \quad \Rightarrow \frac{n^2 + 2 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} + u + \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 12u + 36u^2 + 36u^2 + 6u + n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} =$$

$$= \frac{6un^2 + 18ux + 6u + 72u^2 + 3x + n^2 - 1}{6(6u + 1)}$$

$$\stackrel{(18b)}{\Rightarrow} (n^2 + 2 + 3(x-2))(6u + 1) = 6un^2 + 18ux + 6u + 72u^2 + 3x + n^2 - 1 \Rightarrow$$

$$(18c) \quad \Rightarrow 6un^2 + 18ux - 24u + n^2 + 3x - 4 = 6un^2 + 18ux + 6u + 72u^2 + 3x + n^2 - 1 \Rightarrow$$

$$\Rightarrow 24u^2 + 10u + 1 = 0 \Rightarrow u_{1,2} = \frac{-10 \pm \sqrt{100 - 96}}{48} = \frac{-10 \pm 2}{48} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (18a)-(18c) is contrary!

Case of the (3b),(3c) - (10d) – (10b):

$$(19a) \quad g_x = g_1 + 3(x-1) \stackrel{(3b),(3c)}{=} n^2 + 1 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3b),(3c)}{=} n^2 + 3 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(19a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 2}{6} = v(6u + 1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} \stackrel{(19a)}{\Rightarrow}$$

$$\stackrel{(19a)}{\Rightarrow} \frac{n^2 + 2 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(19b) \quad \Rightarrow \frac{n^2 + 2 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} - u - \frac{n^2 + 3(x-1) + 2 + 6u}{6(6u + 1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 12u + 36u^2 - 36u^2 - 6u - n^2 - 3(x-1) - 2 - 6u}{6(6u + 1)} =$$

$$= \frac{6un^2 + 18ux - 18u - 3x - n^2 + 1}{6(6u + 1)}$$

$$\stackrel{(19b)}{\Rightarrow} (n^2 + 2 + 3(x-2))(6u + 1) = 6un^2 + 18ux - 18u - 3x - n^2 + 1 \Rightarrow$$

$$(19c) \quad \Rightarrow 6un^2 + 18ux - 24u + n^2 + 3x - 4 = 6un^2 + 18ux + 6u + 72u^2 + 3x + n^2 - 1 \Rightarrow$$

$$\Rightarrow 2n^2 + 6x - 5 = 6u \Rightarrow u = \frac{2n^2 + 6x - 5}{6} = \underbrace{\frac{2n^2 - 5}{6}}_{\text{never an integer}} + x$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (19a)-(19c) is contrary!

Case of the (3d),(3e) - (10c) - (10a):

$$(20a) \quad g_x = g_1 + 3(x-1) \stackrel{(3d),(3e)}{=} n^2 + 4 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3d),(3e)}{=} n^2 + 6 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$\stackrel{(20a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 5}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} \stackrel{(20a)}{\Rightarrow}$$

$$\stackrel{(20a)}{\Rightarrow} \frac{n^2 + 5 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(20b) \quad \Rightarrow \frac{n^2 + 5 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} + u + \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 30u - 36u^2 + 36u^2 - 6u + n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux + 3x + n^2 + 2}{6(6u-1)}$$

$$\stackrel{(20b)}{\Rightarrow} (n^2 + 5 + 3(x-2))(6u-1) = 6un^2 + 18ux + 3x + n^2 + 2 \Rightarrow$$

$$(20c) \quad \Rightarrow 6un^2 + 18ux - 6u - n^2 - 3x + 1 = 6un^2 + 18ux + 3x + n^2 + 2 \Rightarrow$$

$$\Rightarrow -2n^2 - 6x - 1 = 6u \Rightarrow u = \frac{-2n^2 - 6x - 1}{6} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (20a)-(20c) is contrary!

Case of the (3d),(3e) - (10c) - (10b):

$$(21a) \quad g_x = g_1 + 3(x-1) \stackrel{(3d),(3e)}{=} n^2 + 4 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3d),(3e)}{=} n^2 + 6 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(21a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 5}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} \stackrel{(21a)}{\Rightarrow}$$

$$\stackrel{(21a)}{\Rightarrow} \frac{n^2 + 5 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(21b) \quad \Rightarrow \frac{n^2 + 5 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} - u - \frac{n^2 + 3(x-1) + 5 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 30u - 36u^2 - 36u^2 + 6u - n^2 - 3(x-1) - 5 + 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux - 72u^2 + 24u - 3x - n^2 - 2}{6(6u-1)}$$

$$\stackrel{(21b)}{\Rightarrow} (n^2 + 5 + 3(x-2))(6u-1) = 6un^2 + 18ux - 72u^2 + 24u - 3x - n^2 - 2 \Rightarrow$$

$$(21c) \quad \Rightarrow 6un^2 + 18ux - 6u - n^2 - 3x + 1 = 6un^2 + 18ux - 72u^2 + 24u - 3x - n^2 - 2 \Rightarrow$$

$$\Rightarrow 72u^2 - 30u = 0 \Rightarrow u(72u - 30) = 0 \Rightarrow u_1 = 0, \quad u_2 = \frac{30}{72}$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (21a)-(21c) is contrary!

Case of the (3d),(3e) - (10d) - (10a):

$$(22a) \quad \begin{aligned} g_x &= g_1 + 3(x-1) \stackrel{(3d),(3e)}{=} n^2 + 4 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1 \\ g_{x+1} &= g_2 + 3(x-2) \stackrel{(3d),(3e)}{=} n^2 + 6 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1 \end{aligned}$$

$$\stackrel{(22a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 5}{6} = v(6u+1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} \stackrel{(22a)}{\Rightarrow}$$

$$\stackrel{(22a)}{\Rightarrow} \frac{n^2 + 5 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(22b) \quad \begin{aligned} \Rightarrow \frac{n^2 + 5 + 3(x-2)}{6} &= 6u \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} + u + \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} = \\ &= \frac{6un^2 + 18u(x-1) + 30u + 36u^2 + 36u^2 + 6u + n^2 + 3x + 2 + 6u}{6(6u+1)} = \\ &= \frac{6un^2 + 18ux - 72u^2 + 24u + 3x + n^2 + 2}{6(6u+1)} \end{aligned}$$

$$\stackrel{(22b)}{\Rightarrow} (n^2 + 5 + 3(x-2))(6u+1) = 6un^2 + 18ux - 72u^2 + 24u + 3x + n^2 + 2 \Rightarrow$$

$$(22c) \quad \Rightarrow 6un^2 + 18ux - 6u + n^2 + 3x - 1 = 6un^2 + 18ux - 72u^2 + 24u - 3x - n^2 - 2 \Rightarrow$$

$$\Rightarrow 72u^2 - 24u - 3 = 0 \Rightarrow 24u^2 - 8u - 1 = 0 \Rightarrow u_{1,2} = \frac{2 \pm \sqrt{10}}{\underbrace{12}_{\text{not an int eger}}}$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (22a)-(22c) is contrary!

Case of the (3d),(3e) - (10d) - (10b):

$$(23a) \quad g_x = g_1 + 3(x-1) \stackrel{(3d),(3e)}{=} n^2 + 4 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3d),(3e)}{=} n^2 + 6 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(23a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 5}{6} = v(6u+1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} \stackrel{(23a)}{\Rightarrow}$$

$$\stackrel{(23a)}{\Rightarrow} \frac{n^2 + 5 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(23b) \quad \Rightarrow \frac{n^2 + 5 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} - u - \frac{n^2 + 3(x-1) + 5 + 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 30u + 36u^2 - 36u^2 - 6u - n^2 - 3x - 2 - 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18ux - 3x - n^2 - 2}{6(6u+1)}$$

$$\stackrel{(23b)}{\Rightarrow} (n^2 + 5 + 3(x-2))(6u+1) = 6un^2 + 18ux - 3x - n^2 - 2 \Rightarrow$$

$$(23c) \quad \Rightarrow 6un^2 + 18ux - 6u + n^2 + 3x - 1 = 6un^2 + 18ux - 3x - n^2 - 2 \Rightarrow$$

$$\Rightarrow -6u + 2n^2 + 6x + 1 = 0 \Rightarrow u = \frac{2n^2 + 6x + 1}{6} = x + \underbrace{\frac{2n^2 + 1}{6}}_{\text{never an integer}}$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (23a)-(23c) is contrary!

Case of the (3f) - (10c) – (10a):

$$(24a) \quad g_x = g_1 + 3(x-1) \stackrel{(3f)}{=} n^2 + 2 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3f)}{=} n^2 + 4 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$\stackrel{(24a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 3}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} \stackrel{(24a)}{\Rightarrow}$$

$$\stackrel{(24a)}{\Rightarrow} \frac{n^2 + 3 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(24b) \quad \Rightarrow \frac{n^2 + 3 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} + u + \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 18u - 36u^2 + 36u^2 - 6u + n^2 + 3x - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux - 12u + 3x + n^2}{6(6u-1)}$$

$$\stackrel{(24b)}{\Rightarrow} (n^2 + 3 + 3(x-2))(6u-1) = 6un^2 + 18ux - 12u + 3x + n^2 \Rightarrow$$

$$(24c) \quad \Rightarrow 6un^2 + 18ux - 18u - n^2 - 3x + 3 = 6un^2 + 18ux - 12u + 3x + n^2 \Rightarrow$$

$$\Rightarrow 6u + 2n^2 + 6x + 6 = 0 \Rightarrow u = -\frac{2n^2 + 6x + 6}{6} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (24a)-(24c) is contrary!

Case of the (3f) - (10c) – (10b):

$$(25a) \quad g_x = g_1 + 3(x-1) \stackrel{(3f)}{=} n^2 + 2 + 3(x-1) \stackrel{(10c)}{=} 6(6uv + u - v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3f)}{=} n^2 + 4 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(25a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 3}{6} = v(6u-1) + u \Rightarrow v = \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} \stackrel{(25a)}{\Rightarrow}$$

$$\stackrel{(25a)}{\Rightarrow} \frac{n^2 + 3 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(25b) \quad \Rightarrow \frac{n^2 + 3 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} - u - \frac{n^2 + 3(x-1) + 3 - 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 18u - 36u^2 - 36u^2 + 6u - n^2 - 3x + 6u}{6(6u-1)} =$$

$$= \frac{6un^2 + 18ux - 72u^2 + 12u - 3x - n^2}{6(6u-1)}$$

$$\stackrel{(25b)}{\Rightarrow} (n^2 + 3 + 3(x-2))(6u-1) = 6un^2 + 18ux - 72u^2 + 12u - 3x - n^2 \Rightarrow$$

$$(25c) \quad \Rightarrow 6un^2 + 18ux - 18u - n^2 - 3x + 3 = 6un^2 + 18ux - 72u^2 + 12u - 3x - n^2 \Rightarrow$$

$$\Rightarrow 72u^2 - 30u - 3 = 0 \Rightarrow 24u^2 - 10u - 1 = 0 \Rightarrow u_{1,2} = \frac{10 \pm 14}{48} < 1$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (25a)-(25c) is contrary!

Case of the (3f) - (10d) – (10a):

$$(26a) \quad g_x = g_1 + 3(x-1) \stackrel{(3f)}{=} n^2 + 2 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3f)}{=} n^2 + 4 + 3(x-2) \stackrel{(10a)}{=} 6(6uv + u + v) + 1$$

$$\stackrel{(26a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 3}{6} = v(6u+1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} \stackrel{(26a)}{\Rightarrow}$$

$$\stackrel{(26a)}{\Rightarrow} \frac{n^2 + 3 + 3(x-2)}{6} = 6uv + u + v \Rightarrow$$

$$(26b) \quad \Rightarrow \frac{n^2 + 3 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} + u + \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 18u + 36u^2 + 36u^2 + 6u + n^2 + 3x + 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18ux + 72u^2 + 12u + 3x + n^2}{6(6u+1)}$$

$$\stackrel{(26b)}{\Rightarrow} (n^2 + 3 + 3(x-2))(6u+1) = 6un^2 + 18ux - 18u + 3x + n^2 - 3 \Rightarrow$$

$$(26c) \quad \Rightarrow 6un^2 + 18ux - 18u + n^2 + 3x - 3 = 6un^2 + 18ux + 72u^2 + 12u + 3x + n^2 \Rightarrow$$

$$\Rightarrow 72u^2 + 30u + 3 = 0 \Rightarrow 24u^2 + 10u + 1 = 0 \Rightarrow u_{1,2} = \frac{-10 \pm 2}{48} < 0$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (26a)-(26c) is contrary!

Case of the (3f) - (10d) – (10b):

$$(27a) \quad g_x = g_1 + 3(x-1) \stackrel{(3f)}{=} n^2 + 2 + 3(x-1) \stackrel{(10d)}{=} 6(6uv - u + v) - 1$$

$$g_{x+1} = g_2 + 3(x-2) \stackrel{(3f)}{=} n^2 + 4 + 3(x-2) \stackrel{(10b)}{=} 6(6uv - u - v) + 1$$

$$\stackrel{(27a)}{\Rightarrow} \frac{n^2 + 3(x-1) + 3}{6} = v(6u+1) - u \Rightarrow v = \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} \stackrel{(27a)}{\Rightarrow}$$

$$\stackrel{(27a)}{\Rightarrow} \frac{n^2 + 3 + 3(x-2)}{6} = 6uv - u - v \Rightarrow$$

$$(27b) \quad \Rightarrow \frac{n^2 + 3 + 3(x-2)}{6} = 6u \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} - u - \frac{n^2 + 3(x-1) + 3 + 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18u(x-1) + 18u + 36u^2 - 36u^2 - 6u - n^2 - 3x - 6u}{6(6u+1)} =$$

$$= \frac{6un^2 + 18ux - 12u - 3x - n^2}{6(6u+1)}$$

$$\stackrel{(27b)}{\Rightarrow} (n^2 + 3 + 3(x-2))(6u+1) = 6un^2 + 18ux - 18u + 3x + n^2 - 3 \Rightarrow$$

$$(27c) \quad \Rightarrow 6un^2 + 18ux - 18u + n^2 + 3x - 3 = 6un^2 + 18ux - 12u - 3x - n^2 \Rightarrow$$

$$\Rightarrow 6u - 2n^2 - 6x + 3 = 0 \Rightarrow u = x + \underbrace{\frac{2n^2 - 3}{6}}_{\text{never an integer}}$$

u is defined as a natural number (see (10a)-(10d)), but the result of calculations (27a)-(27c) is contrary!

We have proved above that there are always t pieces of g_1, \dots, g_t $6z \pm 1$ -type natural numbers in the interval $[n^2, (n+1)^2]$ (see (9a) - (9f)) and only among these can be the prime numbers in this interval.

However, in the above (12a)-(27c) has been proved that the indirect condition for any n natural number is never satisfied. It follows that, for any natural number n , there exists a prime number among the (5) and (7) type g_1, \dots, g_t natural numbers. This is exactly the proof of item (1). This is exactly proof of (1) statement of our Theorem 1.

Q.E.D.

Table 2.

n	n^2	$(n+1)^2$	$n^2 < q < (n+1)^2$
1	1	4	2
2	4	9	5
3	9	16	11
4	16	25	19
5	25	36	31
6	36	49	41
7	49	64	61
8	64	81	67
9	81	100	83
10	100	121	103
11	121	144	139
12	144	169	149
13	169	196	181
...			
100	10.000	10.201	10.069
...			
1.000	1.000.000	1.002.001	1.000.039
...			
10.000	100.000.000	100.020.001	100.000.213
...			

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