Dénes type Symmetric Prime Number theorem
and its application to proof of the Even Goldbach conjecture

Dénes, Tamás mathematician
Email: tdenest@freemail.hu

Abstract

In this paper we prove the theorem according to which each $N \geq 4$ natural number exist $m_N \geq 0$ natural number, so that $p_{N-} = N - m_N$ and $p_{N+} = N + m_N$ are prime numbers (Dénes type Symmetric Prime Number theorem). A direct consequence of this is the proof of the Even Goldbach conjecture, so we can called Goldbach's theorem after 275 years.

----- . -----

Keywords: prime numbers, even Goldbach conjecture
AMS Classification: 11A41

Theorem 1. *(Dénes type Symmetric Prime Number theorem)*

To each $N \geq 4$ natural number exist $m_N \geq 0$ natural number, so that $p_{N-} = N - m_N$ and $p_{N+} = N + m_N$ are prime numbers.

Equivalent formulation of Theorem 1.

To each $N \geq 4$ natural number exist $m_N \geq 0$ natural number, so that $p_{N-} = N - m_N$ and $p_{N+} = N + m_N$ are prime numbers, and N is the arithmetical mean of these prime numbers.

(s0) $$N = \frac{p_{N-} + p_{N+}}{2}$$

PROOF

$N = 4 \Rightarrow m_N = 1 \Rightarrow \quad p_{N-} = 4 - 1 = 3, \quad p_{N+} = 4 + 1 = 5$

$N = 5 \Rightarrow m_N = 2 \Rightarrow \quad p_{N-} = 5 - 2 = 3, \quad p_{N+} = 5 + 2 = 7$

Thus the Theorem is true for $N = 4$ and $N = 5$.

Due to the 1st Theorem in [Dénes 2001], one of the following is fulfilled for any $p_{N-} \rangle 5$ and $p_{N+} \rangle 5$ prime number:

(s1) $$p_{N-} = 6k - 1 \quad and \quad p_{N+} = 6r - 1 \qquad (k=1,2,3,\ldots),\ (r=1,2,3,\ldots)$$

(s2) $$p_{N-} = 6k - 1 \quad and \quad p_{N+} = 6r + 1 \qquad (k=1,2,3,\ldots),\ (r=1,2,3,\ldots)$$

(s3) $$p_{N-} = 6k + 1 \quad and \quad p_{N+} = 6r - 1 \qquad (k=1,2,3,\ldots),\ (r=1,2,3,\ldots)$$

(s4) $$p_{N-} = 6k + 1 \quad and \quad p_{N+} = 6r + 1 \qquad (k=1,2,3,\ldots),\ (r=1,2,3,\ldots)$$

From the relations (s1)-(s4) we get the following connections to N and m_n:

(s5) $$\begin{aligned} (s1) \Rightarrow \quad & p_{N-} = 6k - 1 = N - m_N \Rightarrow \quad m_N = N - 6k + 1 \Rightarrow \\ \Rightarrow \quad & p_{N+} = 6r - 1 = N + N - 6k + 1 \Rightarrow \quad 0 = 2N - 6k - 6r + 2 \Rightarrow \\ \Rightarrow \quad & N = 3(k + r) - 1 \quad \Rightarrow m_N = 3(r - k) \end{aligned}$$

(s6)
$$(s2) \Rightarrow \quad p_{N-} = 6k-1 = N - m_N \Rightarrow \quad m_N = N - 6k + 1 \Rightarrow$$
$$\Rightarrow \quad p_{N+} = 6r+1 = N + N - 6k + 1 \Rightarrow \quad 0 = 2N - 6k - 6r \Rightarrow$$
$$\Rightarrow \quad N = 3(k+r) \quad \Rightarrow m_N = 3(r-k)+1$$

(s7)
$$(s3) \Rightarrow \quad p_{N-} = 6k+1 = N - m_N \Rightarrow \quad m_N = N - 6k - 1 \Rightarrow$$
$$\Rightarrow \quad p_{N+} = 6r-1 = N + N - 6k - 1 \Rightarrow \quad 0 = 2N - 6k - 6r \Rightarrow$$
$$\Rightarrow \quad N = 3(k+r) \quad \Rightarrow m_N = 3(r-k)-1$$

(s8)
$$(s4) \Rightarrow \quad p_{N-} = 6k+1 = N - m_N \Rightarrow \quad m_N = N - 6k - 1 \Rightarrow$$
$$\Rightarrow \quad p_{N+} = 6r+1 = N + N - 6k - 1 \Rightarrow \quad 0 = 2N - 6k - 6r - 2 \Rightarrow$$
$$\Rightarrow \quad N = 3(k+r)+1 \quad \Rightarrow m_N = 3(r-k)$$

It is easy to see that (s5)-(s8) produces the *N=3u-1*, *N=3u*, *N=3u+1* type natural numbers. The missing *N=3u-2* and *N=3u+2* types can be traced back to them as follows:

(s9) *N=3u-2=3u-2-1+1=3(u-1)+1*, which corresponds to (s8), if *k+r=u-1*

(s10) *N=3u+2=3u+2+1-1=3(u+1)-1*, which corresponds to (s5), if *k+r=u+1*

Thus the formulas (s5)-(s8) produces *all N natural numbers*!

Furthermore it must be proven that for any N and m_N, one of the cases (s5)-(s8) will produce p_{N-}, p_{N+} pairs in which both are prime numbers. Some examples of the above are shown in Table 1.

Table 1.

N			***k+r***	***k***	***r***	m_N	p_{N-}	p_{N+}
10	(s8)	N=3(k+r)+1	3	1	2	3(r-k)=**3**	6k+1=7(prime)	6r+1=13(prime)
15	(s6)	N=3(k+r)	5	1	4	3(r-k)+1=10	6k-1=5(prime)	6r+1=25=5x5 (not prime)
			5	2	3	3(r-k)+1=**4**	6k-1=11(prime)	6r+1=19(prime)
15	(s7)	N=3(k+r)	5	1	4	3(r-k)-1=**8**	6k+1=7(prime)	6r-1=23(prime)
			5	2	3	3(r-k)-1=**2**	6k+1=13(prime)	6r-1=17(prime)
82	(s8)	N=3(k+r)+1	27	1	26	3(r-k)=**75**	6k+1=7(prime)	6r+1=157(prime)
99	(s6)	N=3(k+r)	33	1	32	3(r-k)+1=**94**	6k-1=5(prime)	6r+1=193(prime)
			33	2	31	3(r-k)+1=88	6k-1=11(prime)	6r+1=187=11x17 (not prime)
			33	3	30	3(r-k)+1=**82**	6k-1=17(prime)	6r+1=181(prime)
100	(s8)	N=3(k+r)+1	33	3	30	3(r-k)=**81**	6k+1=19(prime)	6r+1=181(prime)
5.689	(s8)	N=3(k+r)+1	1.896	926	970	3(r-k)=**132**	6k+1=5.557 (prime)	6r+1=5.821(prime)
				10	1.886	3(r-k)=**5.628**	6k+1=61(prime)	6r+1=11.317(prime)
8. 956. 732	(s8)	N=3(k+r)+1	2.985.577	1.492.761	1.492.816	3(r-k)=**165**	6k+1=8.956.567 (prime)	6r+1=8.956.897 (prime)

Dénes, Tamás mathematician Email: tdenest@freemail.hu

Continue of the proof (indirect)

Negation of the Theorem 1:

For any $m_N \langle N$ *and* $p_{N-} = N - m_N$, $p_{N+} = N + m_N$ *pair of natural numbers, at most one member is a prime number*. This can be done in three ways:

I. p_{N-} *is prime and* p_{N+} *is composite number*

II. p_{N-} *is composite and* p_{N+} *is prime number*

III. p_{N-} *and* p_{N+} *both composite number*

Case I.

(s11) $\forall m_N \langle N \Rightarrow p_{N-} = N - m_N \ (prime) \Rightarrow p_{N+} = N + m_N = p_{N-} + 2m_N \ (not\ prime)$

Due to the Theorem 2. in [Dénes 2001] for p_{N+} one of the following is fulfilled:

(s12) $p_{N+} = 6r - 1$ and *r=6uv+u-v* (*u=1,2,3, …*), (*v=1,2,3,…*)

(s13) $p_{N+} = 6r - 1$ and *r=6uv-u+v* (*u=1,2,3, …*), (*v=1,2,3,…*)

(s14) $p_{N+} = 6r + 1$ and *r=6uv+u+v* (*u=1,2,3, …*), (*v=1,2,3,…*)

(s15) $p_{N+} = 6r + 1$ and *r=6uv-u-v* (*u=1,2,3, …*), (*v=1,2,3,…*)

As it follows from (s11) and (s12):

(s16)
$$p_{N+} = p_{N-} + 2m_N = 6(6uv + u - v) - 1 \Rightarrow \quad m_N = 3(6uv + u - v) - \frac{p_{N-} + 1}{2} \quad \Rightarrow$$
$$\Rightarrow \quad N = p_{N-} + m_N = p_{N-} + 3(6uv + u - v) - \frac{p_{N-} + 1}{2} = \frac{p_{N-} + 6(6uv + u - v) - 1}{2}$$

As it follows from (s11) and (s13):

(s17)
$$p_{N+} = p_{N-} + 2m_N = 6(6uv - u + v) - 1 \Rightarrow \quad m_N = 3(6uv - u + v) - \frac{p_{N-} + 1}{2} \quad \Rightarrow$$
$$\Rightarrow \quad N = p_{N-} + m_N = p_{N-} + 3(6uv - u + v) - \frac{p_{N-} + 1}{2} = \frac{p_{N-} + 6(6uv - u + v) - 1}{2}$$

As it follows from (s11) and (s14):

(s18)
$$p_{N+} = p_{N-} + 2m_N = 6(6uv + u + v) + 1 \Rightarrow \quad m_N = 3(6uv + u + v) - \frac{p_{N-} - 1}{2} \quad \Rightarrow$$
$$\Rightarrow \quad N = p_{N-} + m_N = p_{N-} + 3(6uv + u + v) - \frac{p_{N-} - 1}{2} = \frac{p_{N-} + 6(6uv + u + v) + 1}{2}$$

As it follows from (s11) and (s15):

(s19)
$$p_{N+} = p_{N-} + 2m_N = 6(6uv - u - v) + 1 \Rightarrow \quad m_N = 3(6uv - u - v) - \frac{p_{N-} - 1}{2} \quad \Rightarrow$$
$$\Rightarrow \quad N = p_{N-} + m_N = p_{N-} + 3(6uv - u - v) - \frac{p_{N-} - 1}{2} = \frac{p_{N-} + 6(6uv - u - v) + 1}{2}$$

Dénes, Tamás mathematician Email: tdenest@freemail.hu

The consequence of the indirect premise is that the statement (s11) for the given N, must be true for every $m_N<N$. So it is sufficient to show that *there is no N to which* all the equations (s16)-(s19) are met. Namely then the following equations (s20), (s22), (s23), (s25), (s26), (s27) must be met:

(s20) $$N \overset{(s16)}{=} \frac{p_{N-}+6(6uv+u-v)-1}{2} \overset{(s17)}{=} \frac{p_{N-}+6(6uv-u+v)-1}{2} \Rightarrow u=v$$

Substituting u=v in the equations (s16)-(s19) for p_{N+} we get the following:

(s21)
$$u=v \overset{(s16),(s17)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3\cdot 6u^2-\frac{p_{N-}+1}{2}\right)=p_{N-}+36u^2-p_{N-}-1=$$
$$=36u^2-1=(6u-1)(6u+1)$$
$$u=v \overset{(s18)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3(6u^2+2u)-\frac{p_{N-}-1}{2}\right)=p_{N-}+36u^2+12u-p_{N-}+1=$$
$$=36u^2+12u+1=(6u+1)^2$$
$$u=v \overset{(s19)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3(6u^2-2u)-\frac{p_{N-}-1}{2}\right)=p_{N-}+36u^2-12u-p_{N-}+1=$$
$$=36u^2-12u+1=(6u-1)^2$$

From (s21) can be seen that if u=v then p_{N+} is never a prime number, namely the indirect condition (s11) is satisfied.

(s22)
$$N \overset{(s16)}{=} \frac{p_{N-}+6(6uv+u-v)-1}{2} \overset{(s18)}{=} \frac{p_{N-}+6(6uv+u+v)+1}{2} \Rightarrow$$
$$\Rightarrow 6(6uv+u-v-6uv-u-v)=2 \Rightarrow v=-\frac{1}{6}$$

v is a natural number under conditions (s12)-(s15) so case (s22) is not possible.

(s23)
$$N \overset{(s16)}{=} \frac{p_{N-}+6(6uv+u-v)-1}{2} \overset{(s19)}{=} \frac{p_{N-}+6(6uv-u-v)+1}{2} \Rightarrow$$
$$\Rightarrow 6(6uv+u-v-6uv+u+v)=2 \Rightarrow u=\frac{1}{6}$$

The $u=\frac{1}{6}$ result substituting to (s16)-(s19) for p_{N+} we get the following:

Dénes, Tamás mathematician Email: tdenest@freemail.hu

(s24)
$$u=\frac{1}{6}\overset{(s16),(s17)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3(v+\frac{1}{6}-v)-\frac{p_{N-}+1}{2}\right)=p_{N-}+1-p_{N-}-1=0$$
$$u=\frac{1}{6}\overset{(s18)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3(v+\frac{1}{6}+v)-\frac{p_{N-}-1}{2}\right)=p_{N-}+12v+1-p_{N-}+1=$$
$$=2(6v+1)$$
$$u=\frac{1}{6}\overset{(s19)}{\Rightarrow} p_{N+}=p_{N-}+2\left(3(v-\frac{1}{6}-v)-\frac{p_{N-}-1}{2}\right)=p_{N-}-1-p_{N-}+1=0$$

(s25)
$$N\overset{(s17)}{=}\frac{p_{N-}+6(6uv-u+v)-1}{2}\overset{(s18)}{=}\frac{p_{N-}+6(6uv+u+v)+1}{2}\Rightarrow$$
$$\Rightarrow 6(6uv-u+v-6uv-u-v)=2\quad\Rightarrow\quad u=-\frac{1}{6}$$

u is a natural number under conditions (s12)-(s15) so case (s25) is not possible.

(s26)
$$N\overset{(s17)}{=}\frac{p_{N-}+6(6uv-u+v)-1}{2}\overset{(s19)}{=}\frac{p_{N-}+6(6uv-u-v)+1}{2}\Rightarrow$$
$$\Rightarrow 6(6uv-u+v-6uv+u+v)=2\quad\Rightarrow\quad v=\frac{1}{6}$$

Since *u* and *v* are symmetrical in terms (s12)-(s15) this case gives the results (s24).

(s27)
$$N\overset{(s18)}{=}\frac{p_{N-}+6(6uv+u+v)+1}{2}\overset{(s19)}{=}\frac{p_{N-}+6(6uv-u-v)+1}{2}\Rightarrow\quad u=-v$$

Under the conditions (s12)-(s15) *u* and *v* are natural numbers, however from the (s27) one of them is in any case negative, so this case can not be produced.

From the formulas (s22)-(s27) is clear that for a given *N*, in case of $u\neq v$ is *not met the (s11) indirect condition for every* m_N , which confirms the statement of the theorem. For example, see the Table 1 line 7.-9. for *N=99*.

Q.E.D.

Case II.

If we apply the formulas (s12)-(s15) to p_{N-}, then in the (s16)-(s27) derivations will be replaced only the sign due to p_{N+}. Thus, the conclusions about p_{N-} are maintained as symmetric pairs of case I.

Dénes, Tamás mathematician Email: tdenest@freemail.hu

Case III.

This case is similar to formula (s11), so it can be described:

(s28) $\forall 1 \leq m_N \leq N-2 \Rightarrow p_{N-} = N - m_N \ (not\ prime) \Rightarrow p_{N+} = N + m_N \ (not\ prime)$

The statement (s28) is summarized in Table 2 below, and thus can only contain composite numbers in columns 2. and 4.

Table 2.

m_N	$p_{N-} = N - m_N$	**N**	$p_{N+} = N + m_N$
1	*N-1*	*N*	*N+1*
2	*N-2*	*N*	*N+2*
3	*N-3*	*N*	*N+3*
4	*N-4*	*N*	*N+4*
…	…	…	…
i	*N-i*	*N*	*N+i*
…	…	…	…
N-5	***5***	*N*	*2N-5*
N-4	*4*	*N*	*2N-4*
N-3	***3***	*N*	*2N-3*
N-2	***2***	*N*	*2N-2*

Since Theorem 1. requires $N \geq 4$, and (s28) from the case III. condition $1 \leq m_N \leq N-2$, therefore for any *N* that fulfills the conditions, Table 2 contain the rows *i=N-2* and *i=N-3*, so the $p_{N-} = 2$ and $p_{N-} = 3$ values associated with them, that is the only even and the first odd prime number. That's enough to refute the statement (s28). This is reinforced by the construction of Table 2, that in column 2. p_{N-} picks up the full values of the closed interval $2 \leq p_{N-} \leq N-1$ so that it takes all prime numbers less than or equal to *N-1*. This, however contradicts the statement (s28), that is, case III. there is no a legal case.

Q.E.D.

Dénes, Tamás mathematician Email: tdenest@freemail.hu

On the above case III. we shown as examples Tables 3. and 4. which contain of *N=6* and *N=10* , where the primes are ***bold***.

Table 3.

m_N	$p_{N-} = N - m_N$	**N**	$p_{N-} = N + m_N$
1	***5***	6	***7***
2	*4*	6	*8*
3	***3***	6	*9*
4	***2***	6	*10*

Table 4.

m_N	$p_{N-} = N - m_N$	**N**	$p_{N-} = N + m_N$
1	*9*	10	***11***
2	*8*	10	*12*
3	***7***	10	***13***
4	*6*	10	*14*
5	***5***	10	*15*
6	*4*	10	*16*
7	***3***	10	***17***
8	***2***	10	*18*

Corollary of the Dénes type Symmetric Prime Number theorem is the proof of the Even Goldbach conjecture

As a consequence of the equivalent formulation of Theorem 1, that for every *N≥4* natural number the following is true:

(s29) $$2N = p_{N-} + p_{N+}$$

Thus all even number are the sum of two prime numbers. This is proven by the strong (even) Goldbach conjecture, which thereafter can be called Goldbach's theorem.[1]

[1] The strong (even) Goldbach conjecture (1742): *Each even number which greater than two is the sum of two prime numbers.*

Dénes, Tamás mathematician Email: tdenest@freemail.hu

Table 2. *(Examples to illustrate of the Dénes type Symmetric Prime Number theorem and the Goldbach theorem)*

2N	***N***	m_N	$p_{N-} = N - m_N$	$p_{N+} = N + m_N$
8	4	1	**3**=4-1	**5**=4+1
10	5	0	**5**=5-0	**5**=5+0
20	10	3	**7**=10-3	**13**=10+3
30	15	2	**13**=15-2	**17**=15+2
32	16	3	**13**=16-3	**19**=16+3
100	50	3	**47**=50-3	**53**=50+3
112	56	3	**53**=56-3	**59**=56+3
202	101	0	**101**=101-0	**101**=101+0
1.000	500	9	**491**=500-9	**509**=500+9
10.000	5.000	81	**4.919**=5.000-81	**5.081**=5.000+81
100.000	50.000	123	**49.877**=50.000-123	**50.123**=50.000+123
1.000.000	500.000	57	**499.943**=500.000-57	**500.057**=500.000+57
8.000.000	4.000.000	237	**3.999.763**=4.000.000-237	**4.000.237**=4.000.000+237
10.000.000	5.000.000	87	**4.999.913**=5.000.000-87	**5.000.087**=5.000.000+87
30.800.000	15.400.000	237	**15.399.763**=15.400.000-237	**15.400.237**=15.400.000+237
30.971.720	15.485.860	3	**15.485.857**=15.485.860-3	**15.485.863**=15.485.860+3

I would like to express my thanks to Professor Kálmán Győry for his support and to the constructive comments of Márton Szikszai's mathematician, who have contributed to completing the proof.

----- . -----

References

[Dénes 2001] Dénes, Tamás: *Complementary prime-sieve*
PUre Mathematics and Applications, Vol.12 (2001), No. 2, pp. 197-207
http://www.titoktan.hu/_raktar/_e_vilagi_gondolatok/PUMA-CPS.htm