PU.M.A. Vol. **13** (2002), No. 3, pp. xxx–xxx

Estimation of the number of twin primes by application of the Complementary Prime Sive

Tamás Dénes
1182. Marosvásárhely u. 13/a.
Budapest, Hungary
e-mail: titoktan@freemail.hu

(Received: September 25, 2002)

Abstract. In the [1] paper we introduced the "Complementary Prime-Sieve" (C.P.S.) which gives necessary and sufficient conditions to generate composite numbers larger than 3 of the forms $6k-1$ and $6k+1$. Here we give a proof of S.W.Golomb's Theorem (see [3]) about the numbers of twin primes, then we deduce an approximate formulae for the $T(N)$ numbers of twin primes in the $1-N$ interval, also based on C.P.S.

Mathematics Subject Classifications (2000). 11A07, 11A41, 11A51

1 Necessary and sufficient condition for the twin prime theorem

Theorem 1 *$p<q$ are twin primes iff $p=6k-1,\ q=6k+1$ where $k=1,2,3,\ldots$*

Proof. According to the Theorem 1. in [1] (Every prime numbers larger than 3 are of the forms $6k+1$ or $6k-1$) there are two options:

(1) $p=6k-1 \Rightarrow q=p+2=6k+1$

(2) $p=6k+1 \Rightarrow q=p+2=6k+3=3(2k+1)$ but this is not a prime.

Consequently, only case (1) is possible. Inversely the proof is trivial.

□

Corollary 1 *(1) $pq=(6k-1)(6k+1)=36k^2-1$ if p,q are twin primes. This implies the Theorem 2 below:*

Theorem 2 *$(36k^2-1)$ has two prime factors iff $6k-1$ and $6k+1$ are twin primes.*

Proof. If $(36k^2-1)$ has exactly two prime factors, then according to (3) these can only be $6k-1$ and $6k+1$. Thus $6k-1$ and $6k+1$ are twin primes. If $6k-1$ and $6k+1$ are twin primes, then also according to (3), $(36k^2-1)$ cannot have other prime factors.

□

Theorem 2 implies the theorem below regarding about the cardinality of twin primes:

The cardinality of twin primes is finite iff there is a K natural number for which, every $k>K$ implies that $(36k^2-1)$ have at least three prime factors. It can be realized only if at least one of $6k-1$ and $6k+1$ is a composite number. According to the Theorem 2 in [1] (C.P.S. theorem) it is possible iff k can be described as one of the following forms:

(4) $k = 6uv + u + v$ or $k = 6uv - u - v$ or $k = 6uv - u + v$ or $k = 6uv + u - v$

With these states we have proved the Theorem 3 below:

Theorem 3 *The cardinality of twin primes is finite iff there is a K natural number for which every $k > K$ can be written into one of the forms* (4)*.*

In this way we have also proved S.W.Golomb's theorem which he introduced in [3] as an E969 open problem :

"A necessary and sufficient condition that there be infinitely many twin primes is there be infinitely many numbers k not of the forms (4)*."*

2 Estimation of the number of twin primes in the interval $(1-N)$

Let us order the natural numbers of the forms $6k-1$ and $6k+1$ according to Table 1. It can be seen easily that Table 1 contains all the natural numbers of the form $6k-1$ in the second column and all the natural numbers of the form $6k+1$ in the third column, so all the prime numbers as well.[1]

According to the Theorem 1 (see above) all the twin primes are in those lines of the table where there are prime numbers in the second and the third column as well. Thus the meaning of Theorem 3 is that in rows after K of Table 1 only one of the numbers can possibly be a prime number. Thus every $k > K$ row index can be put into one of the forms (4).

Now, we will show that in Table 1 there are rows in which there are certainly not any twin primes. Firstly we examine the cases $k = 5r$, $k = 5r+1$, $k = 5r+2$, $k = 5r+3$, $k = 5r+4$, which cases obviously generate all k row indexes.

If $k = 5r$ $(r = 1, 2, 3, \ldots)$, then $6k-1 = 30r-1$ and $6k+1 = 30r+1$. These two sequences contain each fifth row of the Table 1 where there are twin primes for example in the following (see Table 1): $r = 1, 2, 5, 6, \ldots$, $k = 5, 10, 25, 30, \ldots$

If $k = 5r+1$ $(r = 1, 2, 3, \ldots)$, then $6k-1 = 30r+5 = 5(6r+1)$ which is not a prime number, so these rows definitely don't contain twin primes (see Table 1). For example: $r = 1, 2, 3, 4, \ldots$, $k = 6, 11, 16, 21, \ldots$

If $k = 5r+2$ $(r = 0, 1, 2, 3, \ldots)$, then $6k-1 = 30r+11$ and $6k+1 = 30r+13$. These two sequences also come up in each fifth rows of Table 1 starting from the second row, where there are twin primes. For example (see Table 1): $r = 0, 1, 2, 3, 6, \ldots$, $k = 2, 7, 12, 17, 32, \ldots$

[1] This statement is the consequence of the theorem: "All prime numbers are of the forms of $6k-1$ or $6k+1$." (see [1] Theorem 1.)

Table 1:

k	$6k-1$	$6k+1$	
1	5	7	(twin prime)
2	11	13	(twin prime)
3	17	19	(twin prime)
4	23	25	$25=(6\cdot 1-1)(6\cdot 1-1)=5\cdot 5\mapsto u=1,\ v=1$
5	29	31	(twin prime)
6	35	37	$35{=}(6\cdot 1-1)(6\cdot 1+1)=5\cdot 7\mapsto\ u=1,\ v=1$
7	41	43	(twin prime)
8	47	49	$49{=}(6\cdot 1+1)(6\cdot 1+1)=7\cdot 7\ \mapsto\ u=1,\ v=1$
9	53	55	$55=(6\cdot 1-1)(6\cdot 2-1)=5\cdot 11\ \mapsto\ u=1,\ v=2$
10	59	61	(twin prime)
11	65	67	$65=(6\cdot 1-1)(6\cdot 2+1)=5\cdot 13\ \mapsto\ u=1,\ v=2$
12	71	73	(twin prime)
13	77	79	$77=(6\cdot 1+1)(6\cdot 2-1)=7\cdot 11\ \mapsto\ u=1,\ v=2$
14	83	85	$85=(6\cdot 1-1)(6\cdot 3-1)=5\cdot 17\ \mapsto\ u=1,\ v=3$
15	89	91	$91=(6\cdot 1+1)(6\cdot 2+1)=7\cdot 13\ \mapsto\ u=1,\ v=2$
16	95	97	$95=(6\cdot 1-1)(6\cdot 3+1)=5\cdot 19\ \mapsto\ u=1,\ v=3$
17	101	103	(twin prime)
18	107	109	(twin prime)
19	113	115	$115=(6\cdot 1-1)(6\cdot 4-1)=5\cdot 23\ \mapsto\ u=1,\ v=4$
20	119	121	$121=(6\cdot 2-1)(6\cdot 2-1)=11\cdot 11\ \mapsto\ u=2,\ v=2$
$\vdots$			
K	$6k-1$	$6k+1$	
$\vdots$			

If $k=5r+3$ $(r=0,1,2,3,\ldots)$, then $6k-1=30r+17$ and $6k+1=30r+19$. These two sequences also come up in each fifth rows of Table 1 starting from the 3rd row, where there are twin primes. For example (see Table 1): $r=0,3,4,6,\ldots$, $k=3,18,23,33,\ldots$

If $k=5r+4$ $(r=0,1,2,3,\ldots)$, then $6k-1=30r+23$ and $6k+1=30r+25=5(6r+5)$, which is not a prime number, so in these rows there will certainly not be any twin primes.

According to the above mentioned in the rows $k=5r+1$ and $k=5r+4$ of Table 1 there are not any twin primes $(k=4,6,9,11,14,16,\ldots)$ thus

(5) all the twin primes are contained in the $k=5r$, $k=5r+2$, $k=5r+3$ $(k=0,1,2,3,\ldots)$ rows of the Table 1.

Consequently there is an upper bound for the number of twin primes (denote $T(N)$):
The number of rows in Table 1 to N (denote k_N) is: $6k_N+1\le N\ \Rightarrow\ k_N=\left[\frac{N-1}{6}\right]=\left[\frac{N}{6}\right]$.
According to the previous deduction twin primes can occur in maximum $\frac{3}{5}$ part

of these rows:

(6) $T(N) \le \left[\frac{N}{6}\right] \cdot \frac{3}{5} = \left[\frac{N}{10}\right]$.

Let us examine the rows of the forms (5) of Table 1 with the aid of C.P.S. There are definitely not any twin primes in rows (5), if the k rowindex can be put into one of the forms of (4). Because the (4) relations are symmetric for u, v we can examine the cases when $u = \text{constant}$.

(7) $u = 1$ and $k = 5r = 6uv + u + v \Rightarrow k = 7v + 1$ holds, if $v = 5a + 2 \Rightarrow k = 35a + 15$

(8) $u = 1$ and $k = 5r = 6uv - u - v \Rightarrow k = 5v - 1 \Rightarrow 5r = 5v - 1$, it can never be realised

(9) $u = 1$ and $k = 5r = 6uv + u - v \Rightarrow k = 5v + 1 \Rightarrow 5r = 5v + 1$, it can never be realised

(10) $u = 1$ and $k = 5r = 6uv - u + v \Rightarrow k = 7v - 1$ holds, if $v = 5a + 3, \Rightarrow k = 35a + 20$

(11) $u = 1$ and $k = 5r+2 = 6uv+u+v \Rightarrow k = 7v+1$ holds, if $v = 5a+3, \Rightarrow k = 35a + 22$

(12) $u = 1$ and $k = 5r + 2 = 6uv - u - v \Rightarrow k = 5v - 1 \Rightarrow 5r = 5v - 3$, it can never be realised

(13) $u = 1$ and $k = 5r + 2 = 6uv + u - v \Rightarrow k = 5v + 1 \Rightarrow 5r = 5v - 1$, it can never be realised

(14) $u = 1$ and $k = 5r+2 = 6uv-u+v \Rightarrow k = 7v-1$ holds, if $v = 5a+4, \Rightarrow k = 35a + 27$

(15) $u = 1$ and $k = 5r+3 = 6uv+u+v \Rightarrow k = 7v+1$ holds, if $v = 5a+1, \Rightarrow k = 35a + 8$

(16) $u = 1$ and $k = 5r + 3 = 6uv - u - v \Rightarrow k = 5v - 1 \Rightarrow 5r = 5v - 4$, it can never be realised

(17) $u = 1$ and $k = 5r + 3 = 6uv + u - v \Rightarrow k = 5v + 1 \Rightarrow 5r = 5v - 2$, it can never be realised

(18) $u = 1$ and $k = 5r+3 = 6uv-u+v \Rightarrow k = 7v-1$ holds, if $v = 5a+2, \Rightarrow k = 35a + 13$

As a consequence of deductions (7),(10),(11),(14),(15),(18) there are not any twin primes in $\frac{6}{35}$ part of rows (5) containing twin primes. In such a way we can make formulae (6) more precisely:

(19) $T(N) \le \left[\frac{N}{10}\right]\left(1 - \frac{6}{35}\right) = \frac{29}{35}\left[\frac{N}{10}\right]$

The case u=2 leads us to the general formulae.

(20) $u = 2$ and $k = 5r = 6uv + u + v \Rightarrow k = 13v + 2$ holds, if $v = 5a + 1, \Rightarrow k = 65a + 15$

(21) $u = 2$ and $k = 5r = 6uv - u - v \Rightarrow k = 11v - 2$ holds, if $v = 5a + 2, \Rightarrow$ $k = 55a + 20$

(22) $u = 2$ and $k = 5r = 6uv + u - v \Rightarrow k = 11v + 2$ holds, if $v = 5a + 3, \Rightarrow$ $k = 55a + 35$

(23) $u = 2$ and $k = 5r = 6uv - u + v \Rightarrow k = 13v - 2$ holds, if $v = 5a + 4, \Rightarrow$ $k = 65a + 50$

(24) $u = 2$ and $k = 5r + 2 = 6uv + u + v \Rightarrow k = 13v + 2$ holds, if $v = 5a, \Rightarrow$ $k = 65a + 2$

(25) $u = 2$ and $k = 5r + 2 = 6uv - u - v \Rightarrow k = 11v - 2$ holds, if $v = 5a + 4, \Rightarrow k = 55a + 42$

(26) $u = 2$ and $k = 5r + 2 = 6uv + u - v \Rightarrow k = 11v + 2$ holds, if $v = 5a, \Rightarrow$ $k = 55a + 2$

(27) $u = 2$ and $k = 5r + 2 = 6uv - u + v \Rightarrow k = 13v - 2$ holds, if $v = 5a + 3, \Rightarrow k = 65a + 37$

(28) $u = 2$ and $k = 5r + 3 = 6uv + u + v \Rightarrow k = 13v + 2$ holds, if $v = 5a + 2, \Rightarrow k = 65a + 28$

(29) $u = 2$ and $k = 5r + 3 = 6uv - u - v \Rightarrow k = 11v - 2$ holds, if $v = 5a, \Rightarrow$ $k = 55a - 2$

(30) $u = 2$ and $k = 5r + 3 = 6uv + u - v \Rightarrow k = 11v + 2$ holds, if $v = 5a + 1, \Rightarrow k = 55a + 13$

(31) $u = 2$ and $k = 5r + 3 = 6uv - u + v \Rightarrow k = 13v - 2$ holds, if $v = 5a, \Rightarrow$ $k = 65a - 2$

Connections (20)-(31) show that starting from a certain row (which rows are all different) each 55th and 65th rows definitely do not contain twin primes. That is according to (19), $\frac{6}{55} + \frac{6}{65}$ part of the rows containing twin primes potentially fall out as well. We can see that it's true for any u in general that s_u part of the rows containing twin primes potentially definitely do not contain twin primes, where

$$(32) \qquad s_u = \frac{6}{5}\left(\frac{1}{6u+1} + \frac{1}{6u-1}\right)$$

We showed in C.P.S. method that if we want to generate prime numbers until N then we have to "run" u from 1 until $\left[\frac{\sqrt{N}}{6}\right]$. Thus, the number of rows without twin primes below $1 - N$ is:

$$(33) \ \frac{6}{35} + \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} s_u = \frac{6}{5} \cdot \left(\frac{1}{7} + \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \frac{1}{6u+1} + \frac{1}{6u-1}\right)$$

But of the statement explained in C.P.S. method, all the u,v pairs generate k by $m(N)$ multiplicity in $K(N)$ steps. Thus, the number of rows not containing twin primes (denote $S(N)$) is the following:

(34) $S(N) \approx \left(1 - \frac{m(N)}{K(N)}\right) \left(\frac{6}{35} + \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} s_u\right)$

This causes the estimation from (19) to $T(N)$ below:

(35) $T(N) \le \left[\frac{N}{10}\right](1 - S(N)) \approx$

$$\approx \left[\frac{N}{10}\right] \left(1 - \left(1 - \frac{m(N)}{K(N)}\right) \frac{6}{5} \left(\frac{1}{7} + \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \left(\frac{1}{6u+1} + \frac{1}{6u-1}\right)\right)\right)$$

(36) $\sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \left(\frac{1}{6u+1} + \frac{1}{6u-1}\right) = \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \frac{12u}{36u^2-1} \approx \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \frac{12}{36u} = \frac{1}{3} \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \frac{1}{u}$

We use the estimated formulae below (see [2] 2.pp.):

(37) $1 + \log(n) \ \ge \ \sum_{u=1}^{n} \frac{1}{u} \rangle \ \log(n+1) \quad \Rightarrow$

(38) $\sum_{u=1}^{n} \frac{1}{u} \ \approx \ \frac{1+\log(n)+\log(n+1)}{2} = \frac{1+\log(n(n+1))}{2}$

From (38) and (36) the result is the following:

(39) $\frac{1}{3} \sum_{u=2}^{\left[\frac{\sqrt{N}}{6}\right]} \frac{1}{u} \ \approx \ \frac{1}{3} \left(\frac{1+\log\left(\frac{\sqrt{N}}{6}\left(\frac{\sqrt{N}}{6}+1\right)\right)}{2} - 1\right) = \frac{\log\left(\frac{N+6\sqrt{N}}{36}\right)-1}{6}$

(40) $T(N) \ \approx \left[\frac{N}{10}\right] \left(1 - \left(1 - \frac{m(N)}{K(N)}\right) \frac{6}{5} \left(\frac{1}{7} + \frac{\log\left(\frac{N+6\sqrt{N}}{36}\right)-1}{6}\right)\right) =$

$$= \left[\frac{N}{10}\right] \left(1 - \left(1 - \frac{m(N)}{K(N)}\right) \left(\frac{\log(N+6\sqrt{N})}{5} - \underbrace{\frac{1 + 7 \cdot \log 36}{35}}_{0.7452752}\right)\right)$$

(41) $\log(N + 6\sqrt{N}) = \log\sqrt{N}(\sqrt{N} + 6) = \frac{\log N}{2} + \log(\sqrt{N} + 6) \stackrel{N\to\infty}{\longrightarrow} \log N$

(42) $T(N) \approx \left[\frac{N}{10}\right] \left(1 - \left(1 - \frac{m(N)}{K(N)}\right) \left(\frac{\log N}{5} - 0.7452752\right)\right)$

The result of Hardy and Littlewood (see [4]) for the number of twin primes until x is the following (x is an arbitrary natural number and $T(x)$ the number of twin primes until x):

(43) $T(x) \approx C\left(\frac{x}{\log^2 x}\right) \quad \text{where} \quad C = 2\prod_{p>2}\left(1 - \frac{1}{(p-1)^2}\right) = 1.32032...$

where $p > 2$ means the prime numbers are greater than 2 until x.

Let us denote the real number of twin primes until N to $RT(N)$. Than the comparison of the results above are contained in Table 2.

Table 2:

N	$\frac{m(N)}{K(N)}$	$RT(N)$	$T(N)$	$T(N)$ rel.err.%	$T(x)$	$T(x)$ rel.err.%
100	.1	8	8	—	6	25%
1.000	.165	35	42	20%	27	23%
2.000	.23	61	75	22%	45	26%
3.000	.27	81	101	24%	61	25%
4.000	.29	103	131	27%	76	26%
5.000	.3	126	156	23%	90	29%
10.000	.35	205	268	30%	155	24%
30.000	.42	467	660	41%	372	21%
50.000	.445	705	964	36%	563	21%
1.000.000	.56	8169	9774	19%	6915	16%
2.000.000	.58	14871	16102	8%	12541	16%
3.000.000	.59	20933	20734	1%	17803	15%
4.000.000	.6	26861	27451	2%	22847	15%
5.000.000	.605	32464	31398	3%	27739	15%
8.000.000	.618	48619	46103	5%	41797	14%

Acknowledgment. I am happy to thank Dr S.W.Golomb for the informative letter in which he explained that he has published the theorem in [3]: *"A necessary and sufficient condition that there be infinitely many twin primes is that there be infinitely many numbers n not of the form $6ab \pm a \pm b$."* and he wrote up-to-date historical background to it.

References

[1] T.Dénes, *Complementary prime-sive*, PU.M.A. **12** (2002), No. 2. 2002.

[2] P. Erdős, J. Surányi, *Selected chapters from the theory of numbers* (in Hungarian) Polygon, Szeged, 1996.

[3] S.W.Golomb, *Problem E969*, American Math. Monthly, **58** (1951), no.5., 338.

[4] S.W. Golomb, *The Twin Prime Constant*, American Math. Monthly, **67** (1960), no.8., 767–769.

[5] W.Sierpinski, *A Selection of Problems in the Theory of Numbers*, Pergamon Press, Oxford, U.K., 1964.

[6] Maria Suzuki, *Alternative Formulations of the Twin Prime Problem*, The Amer. Math. Monthly, **107** (2000).